Simulation

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Contents

What is simulation?

• Simulation is (at least from the teletraffic point of view) a statistical method to estimate the performance
(or some other important characteristic)
of the system under consideration.

• It typically consists of the following four phases:

– Modelling of the system (real or imaginary) as a dynamic stochastic process

– Generation of the realizations of this stochastic process (“observations”)

  • Such realizations are called simulation runs

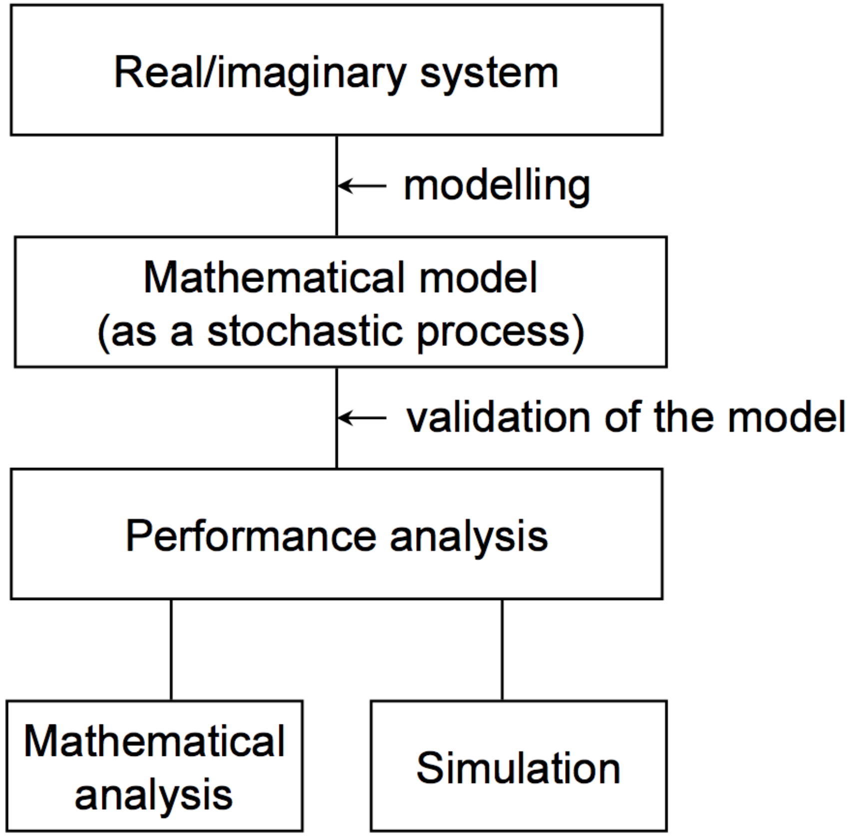
– Collection of data (“measurements”)

– Statistical analysis of the gathered data, and drawing conclusions

Alternative to what?

- In previous lectures, we have looked at an alternative way to determine the performance: mathematical analysis
- We considered the following two phases:
  - – Modelling of the system as a stochastic process.
    (In this course, we have restricted ourselves to birth-death processes.)
  - – Solving of the model by means of mathematical analysis
- The modelling phase is common to both of them (mathematical, simulation approaches)
- However, the accuracy (faithfulness) of the model that these methods allow can be very different
  - – unlike simulation, mathematical analysis typically requires (heavily) restrictive assumptions to be made

Performance analysis of a teletraffic system



Analysis vs. simulation (1)

- Pros of analysis
  - Results produced rapidly (after the analysis is made)
  - Exact (accurate) results (for the model)
  - Gives insight
  - Optimization possible (but typically hard)
- Cons of analysis
  - Requires restrictive assumptions
    ⇒ the resulting model is typically too simple
    (e.g. only stationary behavior)
    ⇒ performance analysis of complicated systems impossible
  - Even under these assumptions, the analysis itself may be (extremely) hard

Analysis vs. simulation (2)

- Pros of simulation
  - No restrictive assumptions needed (in principle)
    - ⇒ performance analysis of complicated systems possible
  - Modelling straightforward
- Cons of simulation
  - Production of results time-consuming
    (simulation programs being typically processor intensive)
  - Results inaccurate (however, they can be made as accurate as required by increasing the number of simulation runs, but this takes even more time)
  - Does not necessarily offer a general insight
  - Optimization possible only between very few alternatives (parameter combinations or controls)

Steps in simulating a stochastic process

- Modelling of the system as a stochastic process
  - This has already been discussed in this course.
  - In the sequel, we will take the model (that is: the stochastic process) for granted.
  - In addition, we will restrict ourselves to simple teletraffic models.
- Generation of the realizations of this stochastic process
  - Generation of random numbers
  - Construction of the realization of the process from event to event (discrete event simulation)
  - Often this step is understood as THE simulation, however this is not generally the case
- Collection of data
  - Transient phase vs. steady state (stationarity, equilibrium)
- Statistical analysis and conclusions
  - Point estimators
  - Confidence intervals

Implementation

- Simulation is typically implemented as a computer program
- Simulation program generally comprises the following phases (excluding modelling and conclusions)
  - Generation of the realizations of the stochastic process
  - Collection of data
  - Statistical analysis of the gathered data
- Simulation program can be implemented by
  - a general-purpose programming language • e.g. C or C++
    - most flexible, but tedious and prone to programming errors
  - utilizing simulation-specific program libraries
    - e.g. CNCL
  - utilizing simulation-specific software
  - e.g. OPNET, BONeS, NS (in part based on p-libraries)
  - most rapid and reliable (depending on the s/w), but rigid

Other simulation types

What we have described above, is a discrete event simulation

- the simulation is discrete (event-based), dynamic (evolving in time) and stochastic (including random components)
- i.e. how to simulate the time evolvement of the mathematical model of the system under consideration, when the aim is to gather information on the system behavior
- We consider only this type of simulation in this lecture

• Other types:

- continuous simulation: state and parameter spaces of the process are continuous; description of the system typically by differential equations, e.g. simulation of the trajectory of an aircraft
- static simulation: time plays no role as there is no process that produces the events, e.g. numerical integration of a multi-dimensional integral by Monte Carlo method
- deterministic simulation: no random components, e.g. the first example above

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Generation of traffic process realizations

- Assume that we have modelled as a stochastic process the evolution of the system
- Next step is to generate realizations of this process.
  - For this, we have to:
    - Generate a realization (value) for all the random variables affecting the evolution of the process (taking properly into account all the (statistical) dependencies between these variables)
    - Construct a realization of the process (using the generated values)
  - These two parts are overlapping, they are not done separately
  - Realizations for random variables are generated by utilizing (pseudo) random number generators
  - The realization of the process is constructed from event to event (discrete event simulation)

Discrete event simulation (1)

- Idea: simulation evolves from event to event
  - If nothing happens during an interval, we can just skip it!
- Basic events modify (somehow) the state of the system
  - e.g. arrivals and departures of customers in a simple teletraffic model
- Extra events related to the data collection
  - including the event for stopping the simulation run or collecting data
- Event identification:
  - occurrence time (when event is handled) and
  - event type (what and how event is handled)

Discrete event simulation (2)

- Events are organized as an event list (aka eminent event list)
  - Events in this list are sorted in ascending order by the occurrence time
    - first: the event occurring next
  - Events are handled one-by-one (in this order) while, at the same time, generating new events to occur later
  - When the event has been processed, it is removed from the list
- Simulation clock tells the occurrence time of the next event
  - progressing by jumps
- System state tells the current state of the system

Discrete event simulation (3)

- General algorithm for a single simulation run:

1 Initialization

simulation clock = 0

system state = given initial value

for each event type, generate next event (whenever possible)

construct the event list from these events

2 Event handling

simulation clock = occurrence time of the next event

handle the event including

– generation of new events and their addition to the event list

– updating of the system state

- delete the event from the event list

3 Stopping test

- if positive, then stop the simulation run; otherwise return to 2

Example1 (1)

- Task: Simulate the M/M/1 queue (more precisely: the evolution of the queue length process) from time 0 to time T assuming that the queue is empty at time 0 and omitting any data collection
  - System state (at time t) = queue length X_t
    - initial value: $X_0 = 0$
  - Basic events:
    - customer arrivals
    - customer departures
  - Extra event:
    - stopping of the simulation run at time T
- Note: No collection of data in this example

Example1 (2)

• Initialization:

– initialize the system state: $X_0 = 0$

– generate the time till the first arrival from the Exp(λ) distribution • Handling of an arrival event (occurring at some time t):

– update the system state: $X_t = X_t + 1$

– if $X_t = 1$, then generate the time $(t + S)$ till the next departure, where S is

from the $Exp(\mu)$ distribution
– generate the time $(t + I)$ till the next arrival, where I is from the $Exp(\lambda)$

distribution

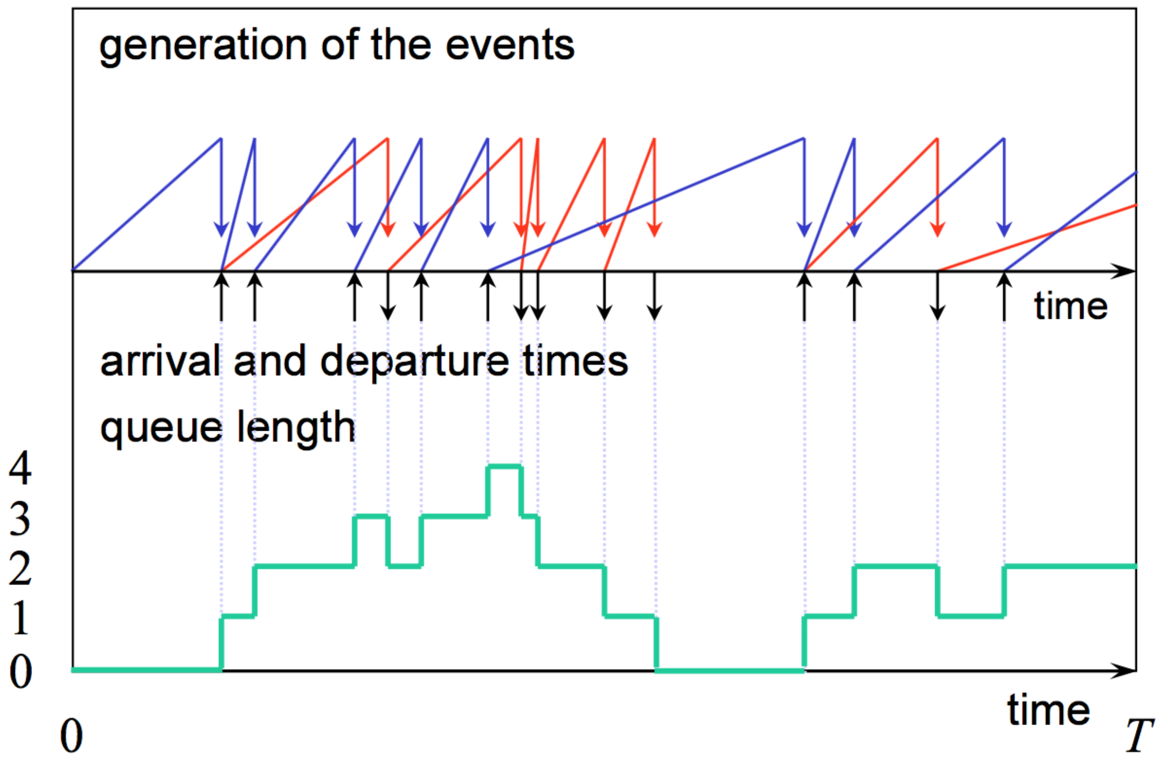
• Handling of a departure event (occuring at some time t):

– update the system state: $X_t = X_t - 1$

– if $X_t > 0$, then generate the time $(t + S)$ till the next departure, where S is

from the $Exp(\mu)$ distribution • Stopping test: $t > T$

Example1 (3)


generation of the events
time
arrival and departure times
queue length
4
3
2
1
0
0
time
T

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- Introduction
- Generation of traffic process realizations
- Generation of random variable realizations
- Collection of data
- Statistical analysis

Generation of random variable realizations

- Based on (pseudo) random number generators
- First step:
  - – generation of independent uniformly distributed random variables between 0 and 1 (i.e. from $U(0,1)$ distribution) by using random number generators
- Step from the U(0,1) distribution to the desired distribution:
  - – rescaling ($\Rightarrow$ $U(a,b)$) (slide 27)
  - – discretization ($\Rightarrow$ $Bernoulli(p), Bin(n,p), Poisson(a), Geom(p)$) (slide 28)
  - – inverse transform ($\Rightarrow$ $Exp(\lambda)$) (slide 29)
  - – other transforms ($\Rightarrow$ $N(0,1)$ $\Rightarrow$ $N(\mu, \sigma 2)$) (slide 31,32)
  - – acceptance-rejection method (for any continuous random variable defined in a finite interval whose density function is bounded)
    - two independent $U(0,1)$ distributed random variables needed

Random number generator

• Random number generator is an algorithm generating (pseudo) random integers Zi in some interval $0,1,\ldots,m\ -1$

– The sequence generated is always periodic (goal: this period should be as long as possible)

– Strictly speaking, the numbers generated are not random at all, but totally predictable (thus: pseudo)

– In practice, however, if the generator is well designed, the numbers “appear” to be IID with uniform distribution inside the set $\{0,1,\ldots,m-1\}$

• Validation of a random number generator can be based on empirical (statistical) and theoretical tests:

– uniformity of the generated empirical distribution

– independence of the generated random numbers (no correlation)

Random number generator types

- Linear congruential generator
  - the simplest one
  - next random number is based on just the current one: $Z_{i+1} = f(Z_i)$ ⇒ period at most m
- Multiplicative congruential generator
  - even simpler
  - a special case of the first type
- Others:
  - Additive congruential generators, shuffling, etc.

Linear congruential generator (LCG)

- Linear congruential generator (LCG) uses the following algorithm to generate random numbers belonging to {0,1,..., m−1}:

$$Z_{i+1} = (aZ_i + c) mod\ m$$

  – Here a, c and m are fixed non-negative integers ($a < m, c < m$)

  – In addition, the starting value (seed) $Z_0 < m$ should be specified

- Remarks:
  – Parameters a, c and m should be chosen with care, otherwise the result can be very poor

  – By a right choice of parameters,
  it is possible to achieve the full period m

    - e.g.$m = 2^b, c\ odd, a = 4k + 1 (b\ often\ 48)$

Multiplicative congruential generator (MCG)

- Multiplicative congruential generator (MCG) uses the following algorithm to generate random numbers belonging to {0,1,..., m−1}:

$$Z_{i+1} = (aZ_i) \bmod m$$

  - Here a and m are fixed non-negative integers (a < m)
  - In addition, the starting value (seed) $Z_0 < m$ should be specified
- Remarks:
  - MCG is clearly a special case of LCG: c = 0
  - Parameters a and m should (still) be chosen with care
  - In this case, it is not possible to achieve the full period m
    - e.g. if $m = 2^b$, then the maximum period is 2^{b-2}
  - However, for m prime, period m−1 is possible (by a proper choice of a)

    PMMLCG = prime modulus multiplicative LCG

    e.g. $m = 2^{31}-1$ and a = 16,807 (or 630,360,016)

U(0,1) distribution

- Let Z denote a (pseudo) random number belonging to $\{0,1,\ldots,m-1\}$
- Then (approximately)

$$\mathrm{U} = \frac{Z}{m} \approx U(0,1)$$

U(a,b) distribution

- Let $U \sim U(0,1)$
- Then

$$X = a + (b-a)U \sim U(a,b)$$

- This is called the rescaling method

Discretization method

- Let $U \sim U(0,1)$
- Assume that Y is a discrete random variable

  – with value set $S = \{0,1,\ldots,n\}$ or $S = \{0,1,2,\ldots\}$

- Denote: $F(x) = P\{Y \leq x\}$, then

  $X = \min\{x \in S | F(x) \geq U\} \sim Y$

- This is called the discretization method

  – a special case of the inverse transform method

- Example: Bernoulli(p) distribution

$$\mathrm{X} = \begin{cases} 0, if\ U \leq 1-p \\ 1, if\ U > 1-p \end{cases} \sim Bernoulli(p)$$

Inverse transform method

- Let $U \sim U(0,1)$
- Assume that Y is a continuous random variable
- Assume further that $F(x) = P\{Y \leq x\}$ is strictly increasing
- Let $F^{-1}(y)$ denote the inverse of the function F(x), then

$X = F^{-1}(U) \sim Y$

- This is called the inverse transform method
- Proof: Since $P\{U \leq u\} = u$ for all $u \in (0,1)$, we have

$$P\{X \leq x\} = P\{F^{-1}(U) \leq x\} = P\{U \leq F(x)\} = F(x)$$

Exp(λ) distribution

- Let $U \sim U(0,1)$
  - Then also $1 - U \sim U(0,1)$
- Let $Y \sim Exp(\lambda)$
  - $F(x) = P\{Y \leq x\} = 1 - e^{-\lambda x}$ is strictly increasing
  - The inverse transform is $F^{-1} = -(1/\lambda)\log(1-y)$
- Thus, by the inverse transform method,

$$X = F^{-1}(1 - \mathrm{U}) = \frac{-1}{\lambda}\log(U) \sim Exp(\lambda)$$

N(0,1) distribution

- Let U1 ~ U(0,1) and U2 ~ U(0,1) be independent
- Then, by so called Box-Müller method,

the following two (transformed) random variables are independent and identically distributed obeying the N(0,1) distribution:

$$X_1 = \sqrt{-2\log(\mathrm{U}_1)}\sin(2\pi U_2) \sim N(0,1)$$
$$X_2 = \sqrt{-2\log(\mathrm{U}_1)}\cos(2\pi U_2) \sim N(0,1)$$

N(μ,σ²) distribution

- Let $X \sim N(0,1)$
- Then, by the rescaling method,

$Y = \mu + \sigma X \sim N(\mu, \sigma^2)$

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Collection of data

- Our starting point was that simulation is needed to estimate the value, say α, of some performance parameter
  - This parameter may be related to the transient or the steady-state behaviour of the system.
  - Examples 1 & 2 (transient phase characteristics)
    - average waiting time of the first k customers in an M/M/1 queue assuming that the system is empty in the beginning
    - average queue length in an M/M/1 queue during the interval [0,T] assuming that the system is empty in the beginning
  - Example 3 (steady-state characteristics)
    - the average waiting time in an M/M/1 queue in equilibrium
- Each simulation run yields one sample, say X, describing somehow the parameter under consideration
- For drawing statistically reliable conclusions,
multiple samples, X1,...,Xn, are needed (preferably IID)

Transient phase characteristics (1)

- Example 1:
  - – Consider e.g. the average waiting time of the first k customers in an M/M/1 queue assuming that the system is empty in the beginning
  - – Each simulation run can be stopped when the kth customer enters the service
  - – The sample X based on a single simulation run is in this case: (work in progress)

$$X = \frac{1}{k}\sum_{i=1}^{k} W_i$$

    - Here Wi = waiting time of the ith customer in this simulation run

- Multiple IID samples, $X_1,\ldots,X_n$, can be generated by the method of independent replications:
  - – multiple independent simulation runs (using independent random numbers)

Transient phase characteristics (2)

- Example 2:
  - Consider e.g. the average queue length in an M/M/1 queue during the interval [0,T] assuming that the system is empty in the beginning
  - Each simulation run can be stopped at time T (that is: simulation clock = T)
  - The sample X based on a single simulation run is in this case: (cycle time average)

$$X = \frac{1}{T}\int_0^T Q(t)dt$$

  - Here Q(t) = queue length at time t in this simulation run
  - Note that this integral is easy to calculate, since Q(t) is piecewise constant
- Multiple IID samples, $X_1,\ldots,X_n$, can again be generated by the method of independent replications

Steady-state characteristics (1)

- Collection of data in a single simulation run is in principle similar to that of transient phase simulations
- Collection of data in a single simulation run can typically (but not always) be done only after a warm-up phase (hiding the transient characteristics) resulting in
  - overhead =“extra simulation”
  - bias in estimation
  - need for determination of a sufficiently long warm-up phase
- Multiple samples, $X_1,\ldots,X_n$, may be generated by the following three methods:
  - independent replications
  - batch means

Steady-state characteristics (2)

- Method of independent replications:
  - multiple independent simulation runs of the same system (using independent random numbers)
  - each simulation run includes the warm-up phase ⇒ inefficiency
  - samples IID ⇒ accuracy
- Method of batch means:
  - one (very) long simulation run divided (artificially) into one warm-up phase and n equal length periods (each of which represents a single simulation run)
  - only one warm-up phase ⇒ efficiency
  - samples only approximately IID ⇒ inaccuracy,
    - choice of n, the larger the better

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Parameter estimation

• As mentioned, our starting point was that simulation is needed to estimate the value, say α, of some performance parameter

• Each simulation run yields a (random) sample, say X_i, describing somehow the parameter under consideration

– Sample Xi is called unbiased if $E[X_i] = \alpha$

• Assuming that the samples Xi are IID with mean α and variance σ^2

– Then the sample average

$$\bar{X}_n := \frac{1}{n}\sum_{i=0}^{n} X_i$$

– is unbiased and consistent estimator of α, since

$$\mathrm{E}[\bar{X}_n] := \frac{1}{n}\sum_{i=0}^{n} E[X_i] = \alpha$$

$$D^2[\bar{X}_n] := \frac{1}{n^2}\sum_{i=0}^{n} D^2[X_i] = \frac{1}{n}\sigma^2 \rightarrow 0 \text{ (as } n\rightarrow\infty)$$

Example

- Consider the average waiting time of the first 25 customers in an M/M/1 queue with load ρ = 0.9 assuming that the system is empty in the beginning
  - Theoretical value: α = 2.12 (non-trivial)
  - Samples Xi from ten simulation runs (n = 10):
    - 1.05,6.44,2.65,0.80,1.51,0.55,2.28,2.82,0.41,1.31
  - Sample average (point estimate for α):

$$\bar{X}_n := \frac{1}{n}\sum_{i=0}^{n} X_i = \frac{1}{10}\ (1.05+6.44+\cdots+1.31)=1.98$$

Confidence interval (1)

- Definition: Interval ($\bar{X}_n$ − y, $\bar{X}_n$ + y) is called the confidence interval for the sample average at confidence level 1 − β if

$$P\{| \bar{X}_n - \alpha | \leq y\} = 1 - \beta$$

– Idea: "with probability 1 − β, the parameter α belongs to this interval"

- Assume then that samples X_i, i = 1,...,n, are IID with unknown mean α but known variance σ^2

- By the Central Limit Theorem (see Lecture 5, Slide 48), for large n,

$$\mathrm{Z} := \frac{\bar{X}_n - a}{\sigma / \sqrt{n}} \approx N(0,1)$$

Confidence interval(2)

- Let z_p denote the p-fractile of the N(0,1) distribution
  - That is : $P\{Z \leq zp\} = p \text{ , where } Z \sim N(0,1)$
  - Example : for $\beta = 5\% (1-\beta = 95\%) \Rightarrow Z_{1-\frac{\beta}{2}} = z_{0.975} \approx 1.96 \approx 2.0$
- Proposition: The confidence interval for the sample average at confidence level 1 − β is

$$\bar{X}_n \pm z_{1-\frac{\beta}{2}} . \frac{\sigma}{\sqrt{n}}$$

- Proof: By definition, we have to show that

$$P\{|\bar{X}_n - \alpha| \leq \bar{X}_n \pm z_{1-\frac{\beta}{2}} . \frac{\sigma}{\sqrt{n}}\} = 1 - \beta$$

$$P\{|\overline{X_n} - \alpha| \leq y\} = 1 - \beta$$

$$\Leftrightarrow P\{\frac{|\bar{X}_n - \alpha|}{\sigma/\sqrt{n}} \leq \frac{y}{\sigma/\sqrt{n}}\} = 1 - \beta$$

$$P\{\frac{-y}{\sigma/\sqrt{n}} \leq \frac{X_n - a}{\sigma/\sqrt{n}} \leq \frac{y}{\sigma/\sqrt{n}}\} = 1 - \beta$$

$$\Leftrightarrow \Phi(\frac{y}{\sigma/\sqrt{n}}) - \Phi(\frac{-y}{\sigma/\sqrt{n}}) = 1 - \beta \qquad [\Phi(x) := P\{Z \leq x\}]$$

$$\Leftrightarrow \Phi(\frac{y}{\sigma/\sqrt{n}}) - (1 - \Phi(\frac{y}{\sigma/\sqrt{n}})) = 1 - \beta \qquad [\Phi(-x) = 1 - \Phi(x)]$$

$$\Leftrightarrow \Phi(\frac{y}{\sigma/\sqrt{n}}) = 1 - \frac{\beta}{2}$$

$$\Leftrightarrow \frac{y}{\sigma/\sqrt{n}} = z_{1-\frac{\beta}{2}}$$

$$\Leftrightarrow y = z_{1-\frac{\beta}{2}} . \frac{\sigma}{\sqrt{n}}$$

Confidence interval (3)

- In general, however, the variance σ^2 is unknown (in addition to the mean α)
- It can be estimated by the sample variance:

$$S_n^2 := \frac{1}{n-1}\sum_{i=1}^{n}(Xi + \overline{X_n})^2 = \frac{1}{n-1}(\sum_{i=1}^{n} X_i^2 - n\,\bar{X}_n^2)$$

- It is possible to prove that the sample variance is an unbiased and consistent estimator of σ^2:

$$E[S_n^2] = \sigma^2$$
$$D^2[S_n^2] \to 0 \; (n \to \infty)$$

Confidence interval (4)

- Assume that samples X_i are IID obeying the $N(\alpha,\sigma^2)$ distribution with unknown mean α and unknown variance σ^2
- Then it is possible to show that

$$T := \frac{\bar{X}_n - \alpha}{S_n/\sqrt{n}} \sim Student(n-1)$$

- Let tn−1,p denote the p-fractile of the Student(n−1) distribution
  - That is: $P\{T \leq tn_{-1,p}\} = p$, where $T \sim Student(n-1)$
  - Example 1: n = 10 and β = 5%, $t_{n-1,1-(\beta/2)} = t_{9,0.975} \approx 2.26 \approx 2.3$
  - Example 2: n = 100 and β = 5%, $t_{n-1,1-(\beta/2)} = t_{99,0.975} \approx 1.98 \approx 2.0$
- Thus, the conf. interval for the sample average at conf. level 1 − β is

$$\bar{X}_n \pm t_{n-1,1-\frac{\beta}{2}} . \frac{S_n}{\sqrt{n}}$$

Example (continued)

• Consider the average waiting time of the first 25 customers in an M/M/1 queue with load ρ = 0.9 assuming that the system is empty in the beginning

- Theoretical value: α = 2.12
- Samples Xi from ten simulation runs (n = 10):
  - 1.05,6.44,2.65,0.80,1.51,0.55,2.28,2.82,0.41,1.31
- Sample average = 1.98 and the square root of the sample variance:

$$S_n = \sqrt{\frac{1}{9}\left((1.05-1.98)^2 + \ldots + (1.31-1.98)^2\right)} = 1.78$$

– So, the confidence interval (that is: interval estimate for α) at confidence level 95% is

$$\bar{X}_n \pm t_{n-1,1-\frac{\beta}{2}} . \frac{S_n}{\sqrt{n}} = 1.98 \pm 2.26 \cdot \frac{1.78}{\sqrt{10}} = 1.98 \pm 1.27 = (0.71,3.25)$$

Observations

- Simulation results become more accurate (that is: the interval estimate for α becomes narrower) when
  - the number n of simulation runs is increased, or
  - the variance σ^2 of each sample is reduced
    - by running longer individual simulation runs
    - variance reduction methods
- Given the desired accuracy for the simulation results, the number of required simulation runs can be determined dynamically

Literature

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- A.M. Law and W. D. Kelton (1982, 1991)
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THE END