Laplace and Z Transforms

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OUTLINE:

- Z-transform:
  - Definition;
  - Properties;
  - Inversion.
- Laplace transform:
  - Definition;
  - Properties;
  - Inversion.

1. Why transforms

Why we are going to consider them separately:

- most problems for those who did not take specific math courses;
- provide a way to analyze queuing systems.

Types of the transforms:

- Laplace transform;
- Z transform;
- Fourier transform;
- ...

How we call transforms:

- just transform (referring to any transform);
- Z-transform: (probability) generating function;
- Laplace transform: Laplace-Stieltjes transform.

Why we are going to use transforms:

- they naturally appear in analysis of queues;
- they simplify the calculation;
- sometimes they are the only tool.

What kind of transforms we are going to consider:

- Laplace transform for continuous RVs;
- Z transform for discrete RVs.

We basically follow:

- L. Kleinrock, ”Queuing systems, Volume I: Theory,” John Wiley & Sons;
- R. Gabel, R. Roberts, ”Signals and linear systems,” John Wiley & Sons;
- Internet, e.g. www.wikipedia.org

2. Z transform

Assume: we are given discrete function defined on RV X , which takes nonnegative values , $X \in \{0, 1, 2, \ldots\}$.

Denote the point probabilities by p_i $$p_i = P\{X = i\} \quad (1)$$

What we want: compress it into a single one such that:

- it passes unchanged through the system;
- we can decompress it.

Do the following:

- tag each value in sequence multiplying by z^i:
  - why z^i: i is unique, thus, z^i is unique for each p_i.
- get a single function depending on z only $G(z)$ (or $G_X(z)$; also $X(z)$ or $\hat{X}(z)$) by summing all terms:

$$G(z) = G_X(z) = \sum_{i=0}^{\infty} p_i z^i = E[z^X] \quad (2)$$

  - which is called z-transform (or generating function or geometric transform).

2. Z transform

Rationale

- A handy way to record all the values $\{p_0, p_1, \ldots\}$; z is a ‘bookkeeping variable’
- Often G(z) can be explicitly calculated (a simple analytical expression)
- When G(z) is given, one can conversely deduce the values $\{p_0, p_1, \ldots\}$
- Some operations on distributions correspond to much simpler operations on the generating functions
- Often simplifies the solution of recursive equations

Condition of existence for z-transform:

- terms in a sequence grow no faster than geometrically;
- meaning that if there is $a > 0$ for which the following holds:

$$\lim_{i\to\infty} \frac{|p_i|}{a^i} = 0 \quad (3)$$

  – for this sequence z-transform is unique.

Analyticity:

- the sum of all terms in p_i must be finite;
    - if so, then $G(z)$ is analytic on a unit circle $|z| \leq 1$;
- in this case we have:

$$G(1) = \sum_{i=0}^{\infty} p_i \quad (4)$$

Note: analyticity means that the function has unique derivative.

1. Getting z-transforms

Delta function $\delta_i = 1, i = 0, \delta_i = 0, i \neq 0$**:**

- since the only one term is non-zero corresponding to $i = 0$ we have:

$$\delta_i \leftrightarrow z^0 = 1 \quad (5)$$

Delta function shifted by k**:** $\delta_{i-k} = 1, i = k, \delta_i = 0, i \neq k$**:**

- since the only one term is non-zero corresponding to $i = k$ we have:

$$\delta_{i-k} \leftrightarrow z^k \quad (6)$$

Unit step function: $u_i = 1, i = 0, 1, \ldots$**:**

- recall that $u_i = 0$ for $i < 0$;
- we have geometric series:

$$u_i \leftrightarrow \sum_{i=0}^{\infty} 1 z^i = \frac{1}{1-z} \quad (7)$$

Geometric series: $p_i = A\alpha^i$**,** $i = 0, 1, \ldots$**:**

- calculate z-transform as follows:

$$G(z) = \sum_{i=0}^{\infty} A\alpha^i z^i = A\sum_{i=0}^{\infty}(\alpha z)^i = \frac{A}{1-\alpha z} \quad (8)$$

- therefore, we have:

$$p_i = A\alpha^i \leftrightarrow \frac{A}{1-\alpha z} \quad (9)$$

- z-transform is analytic for $|z| \leq 1/\alpha$.

Arbitrary sequence: $\{p_0 = -2, p_1 = 0, p_2 = 4, p_3 = -6\}$:

- calculate z-transform as follows:

$$G(z) = \sum_{i=0}^{3} p_i z^i = -2 + 4z^2 - 6z^3 \quad (10)$$

SEQUENCE		z-TRANSFORM
1. $f_n \quad n = 0, 1, 2, \ldots$	$\Leftrightarrow$	$F(z) = \sum_{n=0}^{\infty} f_n z^n$
2. $u_n = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$		1
3. u_{n-k}		z^k
4. $\delta_n = 1 \quad n = 0, 1, 2, \ldots$		$\frac{1}{1 - z}$
5. δ_{n-k}		$\frac{z^k}{1 - z}$
6. $A\alpha^n$		$\frac{A}{1 - \alpha z}$
7. $n\alpha^n$		$\frac{\alpha z}{(1 - \alpha z)^2}$

SEQUENCE	z-TRANSFORM
8. n	$\dfrac{z}{(1-z)^2}$
9. $n^2\alpha^n$	$\dfrac{\alpha z(1+\alpha z)}{(1-\alpha z)^3}$
10. n^2	$\dfrac{z(1+z)}{(1-z)^3}$
11. $(n+1)\alpha^n$	$\dfrac{1}{(1-\alpha z)^2}$
12. $(n+1)$	$\dfrac{1}{(1-z)^2}$
13. $\dfrac{1}{m!}(n+m)(n+m-1)\cdots(n+1)\alpha^n$	$\dfrac{1}{(1-\alpha z)^{m+1}}$
14. $\dfrac{1}{n!}$	e^z

2. Properties of z-transform

Convolution property: Let X and Y be independent random variables with corresponding distributions:

$p_i = \mathrm{P}\{X = i\} > 0 \quad i = 0, 1, \ldots;$

$q_j = \mathrm{P}\{Y = j\} > 0 \quad j = 0, 1, \ldots;$

- denote their transforms by $G_X(z)$ and $G_Y(z)$;
- convolution is defined as follows: $p_i \odot q_i \leftrightarrow \sum_{k=0}^{i} p_{i-k} q_k$ (11)
- derive the transform of the convolution as:

$$p_i \odot q_i \leftrightarrow \sum_{i=0}^{\infty} (p_i \odot q_i) z^i = \sum_{i=0}^{\infty} \sum_{k=0}^{i} p_{i-k} q_k z^{i-k} z^k \qquad (12)$$

- change the summation $\sum_{i=0}^{\infty} \sum_{k=0}^{i} = \sum_{k=0}^{\infty} \sum_{i=k}^{\infty}$ to get as:

$$p_i \odot q_i \leftrightarrow \sum_{k=0}^{\infty} q_k z^k \sum_{i=k}^{\infty} p_{i-k} z^{i-k} = \sum_{k=0}^{\infty} q_k z^k \sum_{m=0}^{\infty} p_m z^m = G_X(z) G_Y(z) \qquad (13)$$

SEQUENCE			z-TRANSFORM
1. f_n	$n = 0, 1, 2, \ldots$	$\Leftrightarrow$	$F(z) = \sum_{n=0}^{\infty} f_n z^n$
2. $af_n + bg_n$			$aF(z) + bG(z)$
3. $a^n f_n$			$F(az)$
4. $f_{n/k}$	$n = 0, k, 2k, \ldots$		$F(z^k)$
5. f_{n+1}			$\frac{1}{z}[F(z) - f_0]$
6. f_{n+k}	$k > 0$		$\frac{F(z)}{z^k} - \sum_{i=1}^{k} z^{i-k-1} f_{i-1}$
7. f_{n-1}			$zF(z)$
8. f_{n-k}	$k > 0$		$z^k F(z)$
9. nf_n			$z\frac{d}{dz}F(z)$

SEQUENCE	z-TRANSFORM
10. $n(n-1)(n-2),\ldots,(n-m+1)f_n$	$z^m \frac{d^m}{dz^m} F(z)$
11. $f_n \circledast g_n$	$F(z)G(z)$
12. $f_n - f_{n-1}$	$(1-z)F(z)$
13. $\sum_{k=0}^{n} f_k \qquad n = 0, 1, 2, \ldots$	$\frac{F(z)}{1-z}$
14. $\frac{\partial}{\partial a} f_n \qquad$ (a is a parameter of f_n)	$\frac{\partial}{\partial a} F(z)$
15. Series sum property	$F(1) = \sum_{n=0}^{\infty} f_n$
16. Alternating sum property	$F(-1) = \sum_{n=0}^{\infty} (-1)^n f$
17. Initial value theorem	$F(0) = f_0$
18. Intermediate value theorem	$\frac{1}{n!} \frac{d^n F(z)}{dz^n}\Big\rvert_{z=0} = f_n$
19. Final value theorem	$\lim_{z \to 1} (1-z)F(z) = f_\infty$

3. Inverting z-transform **Why we need it:**

- sometimes we need to get p_i when we have $G(z)$;
- example: queuing systems, we will see...

Methods to invert transforms: three methods

1- Develop G(z) in a power series, from which the p_i can be identified as the coefficients of the z_i. The coefficients can also be calculated by derivation (this is actually uses intermediate value theorem (property 18):

$$p_i = \frac{1}{i!}\frac{d^i G(z)}{dz^i}\bigg|_{z=0} = \frac{1}{i!}G^{(i)}(0) \qquad (14)$$

– complicated when many terms are required.

2- By inspection: decompose G(z) in parts the inverse transforms of which are known; e.g. the partial fractions (usage of the inversion formula (see, for example, Kleinrock, "Queuing systems, Vol. I"))

3. By a (path) integral on the complex plane

$$p_i = \frac{1}{2\pi i}\oint \frac{G(z)}{z^{i+1}}\,dz$$

path encircling the origin (must be chosen so that the poles of G(z) are outside the path)

Note: all methods are, at least, time-consuming!!!

4. Example: inverting using inspection method

Basis: partial-fraction expansion:

- technique for expressing a rational function of *z* as a sum of simple terms;
- the idea: get elements that are easily invertible;
- possible when *G* (*z*) is rational function of *z*: *G* (*z*) = *N* (*z*)/*D*(*z*);
- possible when the degree of nominator is less than that of denominator (if not, make it so!).

What we want:

- get terms like:

$$A\alpha^i \leftrightarrow \frac{A}{1-\alpha z} \quad , \quad \frac{1}{m!}(i+m)(i+m-1)\dots(i+1)\alpha^i \leftrightarrow \frac{1}{(1-\alpha z)^{m+1}} \qquad (15)$$

What we then use:

- sum of the transforms equals to the transform of the sum:

$$ap_i + bq_i = aGX(z) + bG_Y(z) \qquad (16)$$

Assumptions:

- $D(z)$ in $G(z) = N(z)/D(z)$ is already in factored form:

$$D(z) = \prod_{l=1}^{k} (1 - \alpha_l z)^{m_l} \tag{17}$$

  – lth root is at $1/\alpha_l$ occurring m_l times.

- **Note:** putting $D(z)$ in the factored form can be complicated.

If above is satisfied you may get $F(z)$ in the following form:

In the general form below: lth root is at $1/\alpha_l$ occurring m_l times

$$G(z) = \frac{A_{11}}{(1-\alpha_1 z)^{m_1}} + \frac{A_{12}}{(1-\alpha_1 z)^{m_1-1}} + \cdots + \frac{A_{1m_1}}{(1-\alpha_1 z)} + \frac{A_{21}}{(1-\alpha_2 z)^{m_2}} + \frac{A_{22}}{(1-\alpha_2 z)^{m_2-1}} + \cdots + \frac{A_{2m_2}}{(1-\alpha_2 z)} + \cdots + \frac{A_{k1}}{(1-\alpha_k z)^{m_k}} + \frac{A_{k2}}{(1-\alpha_k z)^{m_k-1}} + \cdots + \frac{A_{km_k}}{(1-\alpha_k z)} \quad (18)$$

where coefficients are given by

$$A_{lj} = \frac{1}{(j-1)!}\left(-\frac{1}{\alpha_l}\right)^{j-1} \frac{d^{j-1}}{dz^{j-1}}\left((1-\alpha_l z)^{m_l}\frac{N(z)}{D(z)}\right)\Bigg|_{z=1/\alpha_l} \quad (19)$$

Multiplying by $(1-\alpha_l z)^{m_l}$ discards multi- root z= $1/\alpha_l$ in denominator and thus the expression at z= $1/\alpha_l$ is unambiguous

Example:

$$G(z) = \left(\frac{4z^2(1-8z)}{(1-4z)(1-2z)^2}\right) \quad (20)$$

Do the following:

- observe that denominator and nominator have the same degree (i.e. 3);
  - we have to put it in a proper form (degree of nominator must be strictly less);
  - to do so factor out two powers of z to get:

$$G(z) = z^2\left(\frac{4(1-8z)}{(1-4z)(1-2z)^2}\right) \quad (21)$$

- denote the rest by $R(z)$:

$$R(z) = \frac{4(1-8z)}{(1-4z)(1-2z)^2} \quad (22)$$

  - there are three poles of denominator: single pole $z = 1/4$ and double pole $z = 1/2$;
  - we have $k = 2$, $\alpha_1 = 4$, $m_1 = 1$, $\alpha_2 = 2$, $m_2 = 2$.

- now we can rewrite $R(z) = [4(1-8z)]/[(1-4z)(1-2z)^2]$ as

$$R(z) = \frac{4(1-8z)}{(1-4z)(1-2z)^2} = \frac{A_{11}}{1-4z} + \frac{A_{21}}{(1-2z)^2} + \frac{A_{22}}{1-2z} \tag{23}$$

- get elements A_{11}, A_{21} and A_{22} as follows:

$$A_{11} = (1-4z)Q(z)\Big|_{z=\frac{1}{4}} = \frac{4\left(1-\left(\frac{8}{4}\right)\right)}{\left(1-\left(\frac{2}{4}\right)\right)^2} = -16$$

$$A_{21} = (1-2z)^2\, R(z)\Big|_{z=\frac{1}{2}} = \frac{4\left(1-\left(\frac{8}{2}\right)\right)}{(1-(4/2))} = 12 \tag{24}$$

$$A_{22} = -\frac{1}{2}\frac{d}{dz}(1-2z)^2\, R(z)\Big|_{z=\frac{1}{2}} = -\frac{1}{2}\frac{d}{dz}\frac{4(1-8z)}{(1-4z)}\Big|_{z=\frac{1}{2}} = -\frac{1}{2}\,\frac{(1-4z)(-32)-4(1-8z)(-4)}{(1-4z)^2} = 8 \tag{25}$$

- we get the following expression for $R(z)$:

$$R(z) = -\frac{16}{1-4z} + \frac{12}{(1-2z)^2} + \frac{8}{1-2z}$$

- check that you got the same as initially had (place terms under common denominator);
- now we can invert $R(z)$ by inspection:
  - first and third terms are in the form: $A\,\alpha^i \Leftrightarrow A/(1-\alpha z)$;

$$-\frac{16}{1-4z} \Leftrightarrow -16(4)^i$$

$$\frac{8}{1-2z} \Leftrightarrow 8(2)^i$$

  - second term is in the form: $(1/m!)(i+m)(i+m-1)\ldots(i+1)\,\alpha^i \Leftrightarrow 1/(1-\alpha z)^{m+1}$;

$$\frac{12}{(1-2z)^2} \Leftrightarrow 12(i+1)(2)^i$$

– using the linearity $ag_i + bq_i = aG_X(z) + bG_Y(z)$ we get:

$$R(z) \leftrightarrow q_i = \begin{cases} 0 & i < 0 \\ -16(4)^i + 12(i+1)(2)^i + 8(2)^i & i = 0,1,\ldots \end{cases} \quad (26)$$

– using property 8 we take into account factor z^2 in $R(z)$:

$$p_i = -16(4)^{i-2} + 12(i-1)(2)^{i-2} + 8(2)^{i-2} \quad i = 2,3,\ldots \quad (27)$$

– finally optimizing the expression we have for p_i:

$$\begin{aligned} p_i &= 0, && i < 2, \\ p_i &= (2i-1)(2)^i - (4)^i && i = 2,3,\ldots \end{aligned} \quad (28)$$

Notes: other examples are in detail in R. Gabel, R. Roberts, ”Signals and linear systems”.

More Examples:

$$G(z) = \frac{1}{1 - z^2} = 1 + z^2 + z^4 + \cdots$$

$$\Rightarrow \qquad p_i = \begin{cases} 1 & for\ i\ even \\ 0 & for\ i\ odd \end{cases}$$

Example 2

$$G(z) = \frac{1}{(1 - z)(2 - z)} = \frac{2}{1 - z} - \frac{2}{2 - z} = \frac{2}{1 - z} - \frac{1}{1 - z/2}$$

Since $\frac{A}{1 - az}$ corresponds to sequence $A \cdot a^i$ we deduce

$$p_i = 2\,.\,(1)^i - 1\,.\left(\frac{1}{2}\right)^i = 2 - \left(\frac{1}{2}\right)^i$$

3. The Laplace transform

Assume: we are given continuous function $f(t)$ defined on nonzero values:

$$f(t) = 0, t < 0 \qquad (29)$$

What we want: compress it into a single one such that:

- it passes unchanged through the system;
- we can decompress it.

Do the following:

- tag each value of $f(t)$ multiplying by e^{-st}:
  - why e^{-st}: t is unique, thus, e^{-st} is unique for each $f(t)$;
  - why e^{-st}: exponentials pass through linear time-invariant systems unchanged.
- get a single function by integrating over all non-zero values:

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt \qquad (30)$$

  - which gives **two-sided** Laplace transform.

Our case: since $f(t)$ defined on nonzero values we have:

$$F(s) = \int_0^\infty f(t)e^{-st}dt \qquad (31)$$

- which gives **one-sided** Laplace transform (0 means 0^- which means $0 - \epsilon$ for $\epsilon > 0$, $\epsilon \to 0$).

Condition of existence for Laplace transform:

- terms in a sequence must grow no faster than exponential;
- meaning that if there is real number σ_a for which the following holds:

$$\lim_{\tau\to\infty} \int_0^\tau |f(t)|e^{\sigma_a t}dt < 0 \qquad (32)$$

  – Laplace transform exists and unique.

Analyticity of the Laplace transform:

- the integral of $f(t)$ must be finite;
- if so, then $F(s)$ is analytic on a right hand plane of $Re(s) \geq 0$:

$$F(0) = \int_0^\infty f(t)dt \qquad (33)$$

3.1. Getting Laplace transform

Example: one sided exponential function:

$$f(t) = \begin{cases} Ae^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{34}$$

- get the Laplace transform as follows

$$f(t) \leftrightarrow F(s) = \int_0^\infty Ae^{-at}e^{-st} = A\int_0^\infty e^{-(a+s)t}\,dt = \frac{A}{s+a} \quad . \tag{35}$$

Example: unit step function:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{36}$$

- consider it as a special case of one-sided exponential function to get:

$$u(t) \leftrightarrow F(s) = \frac{1}{s} \tag{37}$$

FUNCTION		TRANSFORM
1. $f(t) \quad t \geq 0$	$\Leftrightarrow$	$F^*(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt$
2. $u_0(t)$ (unit impulse)		1
3. $u_0(t-a)$		e^{-as}
4. $u_n(t) \triangleq \frac{d}{dt} u_{n-1}(t)$		s^n
5. $u_{-1}(t) \triangleq \delta(t)$ (unit step)		$\frac{1}{s}$

FUNCTION	TRANSFORM
6. $u_{-1}(t-a)$	$\dfrac{e^{-as}}{s}$
7. $u_{-n}(t) \triangleq \dfrac{t^{n-1}}{(n-1)!}$	$\dfrac{1}{s^n}$
8. $Ae^{-at}\,\delta(t)$	$\dfrac{A}{s+a}$
9. $te^{-at}\,\delta(t)$	$\dfrac{1}{(s+a)^2}$
10. $\dfrac{t^n}{n!}e^{-at}\,\delta(t)$	$\dfrac{1}{(s+a)^{n+1}}$

2. Properties of the Laplace transform

Convolution property:

- consider $f(t) > 0$, $g(t) > 0$ for $t \geq 0$ only;
- denote their transforms by $F(s)$ and $G(s)$;
- convolution is defined as follows: (38)

$$f(t) \odot g(t) \leftrightarrow \int_{-\infty}^{\infty} f(t-x)g(x)dx$$

  – in our case the lower limit is 0^-, the upper limit is ∞.

- derive the transform of the convolution as: $\int_{t=0}^{\infty}\int_{x=0}^{t} = \int_{x=0}^{\infty}\int_{t=x}^{\infty}$

$$f(t) \odot g(t) \leftrightarrow \int_{t=0}^{\infty} (f(t) \odot g(t))e^{-st}dt = \int_{t=0}^{\infty}\int_{x=0}^{t} f(t-x)g(x)dx\, e^{-st}dt$$

$$= \int_{t=0}^{\infty}\int_{x=0}^{t} f(t-x)e^{-s(t-x)}dt\, g(x)e^{-sx}dx = \int_{x=0}^{\infty}\int_{t=x}^{\infty} f(t-x)e^{-s(t-x)}dt\, g(x)e^{-sx}dx \quad (39)$$

$$= \int_{x=0}^{\infty} g(x)e^{-sx}dx \int_{v=0}^{\infty} f(v)e^{-sv}dv = F(s)G(s)$$

FUNCTION		TRANSFORM
1. $f(t) \quad t \geq 0$	$\Leftrightarrow$	$F^*(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt$
2. $af(t) + bg(t)$		$aF^*(s) + bG^*(s)$
3. $f\left(\frac{t}{a}\right) \quad (a > 0)$		$aF^*(as)$
4. $f(t-a)$		$e^{-as}F^*(s)$
5. $e^{-at}f(t)$		$F^*(s+a)$
6. $tf(t)$		$-\frac{dF^*(s)}{ds}$
7. $t^n f(t)$		$(-1)^n \frac{d^n F^*(s)}{ds^n}$
8. $\frac{f(t)}{t}$		$\int_{s_1=s}^{\infty} F^*(s_1)\,ds_1$
9. $\frac{f(t)}{t^n}$		$\int_{s_1=s}^{\infty} ds_1 \int_{s_2=s_1}^{\infty} ds_2 \cdots \int_{s_n=s_{n-1}}^{\infty} ds_n F^*(s_n$

FUNCTION	TRANSFORM
10. $f(t) \circledast g(t)$	$F^*(s)G^*(s)$
11.† $\dfrac{df(t)}{dt}$	$sF^*(s)$
12.† $\dfrac{d^n f(t)}{dt^n}$	$s^n F^*(s)$
13.† $\int_{-\infty}^{t} f(t)\,dt$	$\dfrac{F^*(s)}{s}$
14.† $\underbrace{\int_{-\infty}^{t} \cdots \int_{-\infty}^{t}}_{n \text{ times}} f(t)(dt)^n$	$\dfrac{F^*(s)}{s^n}$
15. $\dfrac{\partial}{\partial a} f(t)$ [a is a parameter]	$\dfrac{\partial}{\partial a} F(s)$
16. Integral property	$F^*(0) = \int_{0^-}^{\infty} f(t)\,dt$
17. Initial value theorem	$\lim_{s\to\infty} sF^*(s) = \lim_{t\to 0} f(t)$
18. Final value theorem	$\lim_{s\to 0} sF^*(s) = \lim_{t\to\infty} f(t)$ if $sF^*(s)$ is analytic for Re $(s) \geq 0$

3.3. Two-sided Laplace transform

If $f(t)$ may the nonzero anywhere on the axis:

$$f(t) \leftrightarrow F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt \tag{40}$$

- define the following functions:

$$f_-(t) = \begin{cases} f(t) & t < 0 \\ 0 & t \geq 0 \end{cases}, \qquad f_+(t) = \begin{cases} 0 & t < 0 \\ f(t) & t \geq 0 \end{cases}. \tag{41}$$

- one may get Laplace transform as follows:

$$f(t) = f_-(t) + f_+(t) \tag{42}$$

- we have the following property:

$$F(s) = F_-(-s) + F_+(s), \quad f_-(t) \leftrightarrow F_-(-s), \quad f_+(t) \leftrightarrow F_+(s) \tag{43}$$

4. Inverting Laplace transforms

There are the following methods:

- inspection method;
- formal inversion integral method.

Inspection method:

- use partial-fraction expansion to:
  - rewrite $F(s)$ as a sum of terms;
  - each term should be recognizable as a transform pair.
- use linearity property to:
  - invert the transform term by term;
  - sum the result to recover $f(t)$.

Note: we have to ensure that $F(s)$ is a rational function of s and can be written as:

$$F(s) = N(s)/D(s) \tag{44}$$

Do the following:

- ensure that the degree of the nominator is less than that of denominator:
  - if this is not the case, make it so;
  - to do so divide $N(s)$ by $D(s)$ until the remainder is less than the degree of $D(s)$;
  - partial-fraction expansion must be carried out for remainder;
  - powers of s can be taken into account using transform 4 (see table).

(45)

- $D(s)$ in $F(s) = N(s)/D(s)$ is already in factored form:

$$D(s) = \prod_{i=1}^{k} (s + \alpha_i)^{m_i}$$

  - ith root is at $1/\alpha_i$ occurring m_i times.
- **note:** putting $D(s)$ in the factored form can be complicated.

If the above satisfied:

- rewrite $F(s)$ as follows:

$$F(s) = \frac{B_{11}}{(s+\alpha_1)^{m_1}} + \frac{B_{12}}{(s+\alpha_1)^{m_1-1}} + \cdots + \frac{B_{1m_1}}{(s+\alpha_1)}$$
$$+ \frac{B_{21}}{(s+\alpha_2)^{m_2}} + \frac{B_{22}}{(s+\alpha_2)^{m_2-1}} + \cdots + \frac{B_{2m_2}}{(s+\alpha_2)}$$
$$+ \cdots$$
$$+ \frac{B_{k1}}{(s+\alpha_k)^{m_k}} + \frac{B_{k2}}{(s+\alpha_k)^{m_k-1}} + \cdots + \frac{B_{km_k}}{(s+\alpha_k)} \qquad (46)$$

- coefficients are given by

$$B_{ij} = \frac{1}{(j-1)!} \frac{d^{j-1}}{ds^{j-1}} \left((s+\alpha_i)^{mi} \frac{N(s)}{D(s)}\right|_{s=-\alpha_i} \qquad (47)$$

Example:

$$F(s) = \frac{8(s^2 + 3s + 1)}{(s+3)(s+1)^3} \quad . \tag{48}$$

- the denominator is already in factored form;
- the degree of the denominator (4) is greater than that of the nominator (2);
- we have $k = 2$, $\alpha_1 = 3$, $m_1 = 1$, $\alpha_2 = 1$, $m_2 = 3$;
- we write $F(s)$ as:

$$F(s) = \frac{B_{11}}{s+3} + \frac{B_{21}}{(s+1)^3} + \frac{B_{22}}{(s+1)^2} + \frac{B_{23}}{s+1} \tag{49}$$

- it is easy to derive B_{11} and B_{21}:

$$B_{11} = (s+3)F(s)|_{s=-3} = 8\ \frac{9-9+1}{(-2)^3} = -1 \tag{50}$$

$$B_{21} = (s+1)^3 F(s)\Big|_{s=-1} = 8\ \frac{1-3+1}{2} = -4 \tag{51}$$

- derive B_{22} differentiating as follows:

$$B_{22} = \frac{d}{ds}\,\frac{8(s^2+3s+1)}{(s+3)}\bigg|_{s=-1}$$
$$= \frac{8(s+3)(2s+3)-(s^2+3s+1)(1)}{(s+3)^2}\bigg|_{s=-1}$$
$$= \frac{8(s^2+6s+8)}{(s+3)^2}\bigg|_{s=-1} = 8\,\frac{1-6+8}{s^2} = 6 \qquad (52)$$

- derive B_{23} differentiating B_{22} once more (what we had prior to evaluation at $s = -1$):

$$B_{23} = \frac{1}{2!}\frac{d^2}{ds^2}\left(\frac{8(s^2+3s+1)}{(s+3)}\right)\bigg|_{s=-1} = \frac{1}{2}\,8\,\frac{d}{ds}\left(\frac{(s^2+6s+8)}{(s+3)^2}\right)\bigg|_{s=-1}$$
$$= 4\,\frac{(s+3)^2(2s+6)-(s^2+6s+8)(s+3)}{(s+3)^4}\bigg|_{s=-1} = 4\,\frac{2^2 4-(1-6+8)(2)(2)}{2^4} = 1 \qquad (53)$$

- finally, we have the following expression for $F(s)$: (54)

$$F(s) = \frac{-1}{s+3} + \frac{-4}{(s+1)^3} + \frac{6}{(s+1)^2} + \frac{1}{s+1}$$

- finally, we have after inversion:

$$f(t) = -e^{-3t} - 2t^2e^{-t} + 6e^{-t} + e^{-t} \quad t \geq 0 \tag{55}$$

$$f(t) = 0 \quad t < 0$$

Checking for errors when doing partial-fraction expansion:

- once we have partial-fraction expansion:
  – combine terms and compare to initial expression for $F(s)$.
- once we get $f(t)$:
  – try to get Laplace transform and compare to $F(s)$.

Notes: other examples are in detail in R. Gabel, R. Roberts, ”Signals and linear systems”.

1- Moment GF

$$M_X(\theta) = E\left[e^{\theta X}\right] = \int_{-\infty}^{\infty} e^{\theta X} dF_X(x)$$

- n-th moment $E[x^n] = m_X^{(n)}(0)$

2- Probability GF

$$P_X(z) \triangleq E(z^X) = \sum_{k=0}^{\infty} P_k z^k \quad |z| \leq 1$$

- Tail GF Q(z)= $\frac{1-P(z)}{1-z}$ $P_X(x)$>k

3- Laplace Transform of non-negative RV

$$\phi_X(s) = E[e^{-sX}] = \int_0^{\infty} f_X(x) e^{-sx} dx$$

- n-th moment

$$E(X^n) = (-1)^n \phi_X^{(n)}(0)$$

4- Characteristic function (Fourier –Stieltjes $F_X(x)$)

$$\phi_X(\omega) = E\left[e^{j\omega X}\right] = \int_{-\infty}^{\infty} e^{j\omega x} dF_X(\omega) \quad -\infty < \omega < \infty$$

Relating Moment Generating function/ Characteristic function

$$\phi_X(\omega) = M_X(\theta)\,|_{\theta=i\omega}$$

Relating Probability Generating function/ Moment Generating function

Using $e^{\theta} = z$ and $g_X(z) = E(z^X)$

$$g_X(e^{\theta}) = E[e^{\theta X}] = M_X(\theta)$$

Using $\theta = \ln(z)$ and $g_X(z) = E(z^X)$

$$M_X(\theta)|_{\ln(z)} = M_X(\ln z) = E(e^{X \ln z}) = E\left(e^{\ln z^X}\right) = E(z^X) = g_X(z)$$