Function of Random Variables

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5. Functions of a Random Variable

Let X be a r.v defined on the model $(\Omega, F, P),$ and suppose $g(x)$ is a function of the variable x. Define

$$Y = g(X). \qquad (5\text{-}1)$$

Is Y necessarily a r.v? If so what is its PDF $F_Y(y),$ pdf $f_Y(y)$?

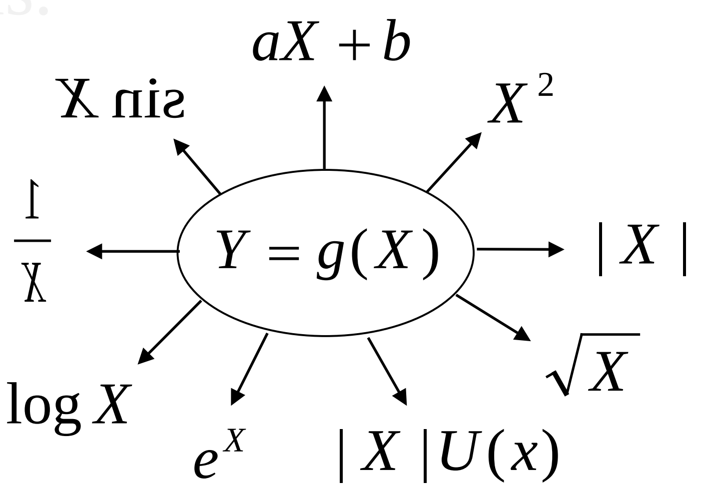
Clearly if Y is a r.v, then for every Borel set B, the set of for which must belong to F. Given that X is a r.v, this is assured if is also a Borel set, i.e., if $g(x)$ is a Borel function. In that case if X is a r.v, so is Y, and for every Borel set B

(5-2)

In particular

$$F_Y(y) = P(Y(\xi) \le y) = P\big(g(X(\xi)) \le y\big) = P\big(X(\xi) \le g^{-1}(-\infty, y]\big). \quad \text{(5-3)}$$

Thus the distribution function as well of the density function of Y can be determined in terms of that of X. To obtain the distribution function of Y, we must determine the Borel set on the x-axis such that for every given y, and the probability of that set. At this point, we shall consider some of the following functions to illustrate the technical details.



Special case:

Theorem: Suppose that g(X) is a function of a random variable X, & the probability mass function of X is $p_x(x)$. Then the expected value of g(X) is

$$E[g(X)] = \sum_{x} g(x) p_{x}(x)$$

Example: Suppose $Y = X^2$ & the distribution of X is as given below. Determine the mean of g(X) by using

1. the definition of expected value, &
2. the previous theorem.

x	p(x)
-2	0.1
-1	0.2
1	0.3
2	0.4

Example: Suppose $Y = X^2$ & the distribution of X is as given below. Determine the mean of g(X) by using

1. ***the definition of expected value***, &
2. the previous theorem.

x	p(x)	y	p(y)
-2	0.1		
-1	0.2		
1	0.3		
2	0.4		

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x	p(x)
-2	0.1
-1	0.2
1	0.3
2	0.4

y	p(y)
1	0.5

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x	p(x)
-2	0.1
-1	0.2
1	0.3
2	0.4

y	p(y)
1	0.5
4	0.5

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x	p(x)
-2	0.1
-1	0.2
1	0.3
2	0.4

y	p(y)	yp(y)
1	0.5	0.5
4	0.5	2.0

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1. ***the definition of expected value***, &
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x	p(x)
-2	0.1
-1	0.2
1	0.3
2	0.4

y	p(y)	yp(y)
1	0.5	0.5
4	0.5	2.0
	E(Y) =	2.5

Example: Suppose $Y = X^2$ & the distribution of X is as given below. Determine the mean of g(X) by using

1. the definition of expected value, &
2. ***the previous theorem***.

x	p(x)	y
-2	0.1	4
-1	0.2	1
1	0.3	1
2	0.4	4

Example: Suppose $Y = X^2$ & the distribution of X is as given below. Determine the mean of g(X) by using

1. the definition of expected value, &
2. ***the previous theorem***.

x	p(x)	y	$yp_X(x)$
-2	0.1	4	0.4
-1	0.2	1	0.2
1	0.3	1	0.3
2	0.4	4	1.6

Example: Suppose $Y = X^2$ & the distribution of X is as given below. Determine the mean of g(X) by using
1. the definition of expected value, &
2. ***the previous theorem***.

x	$p(x)$	y	$yp_X(x)$
-2	0.1	4	0.4
-1	0.2	1	0.2
1	0.3	1	0.3
2	0.4	4	1.6
		E(Y) =	2.5

General case:

Now, we would like to find the distribution of $Y=g(X)$

Method 1

Example 5.1: $Y = aX + b$ (5-4)

Solution: Suppose $a > 0$.

$$F_Y(y) = P\big(Y(\xi) \le y\big) = P\big(aX(\xi) + b \le y\big) = P\left(X(\xi) \le \frac{y-b}{a} \right) = F_X\left(\frac{y-b}{a} \right). \quad (5\text{-}5)$$

and

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a} \right). \quad (5\text{-}6)$$

On the other hand if $a < 0$, then

$$F_Y(y) = P\big(Y(\xi) \le y\big) = P\big(aX(\xi) + b \le y\big) = P\left(X(\xi) > \frac{y-b}{a} \right)$$

$$= 1 - F_X\left(\frac{y-b}{a} \right), \quad (5\text{-}7)$$

and hence

$$f_Y(y) = -\frac{1}{a} f_X\left(\frac{y-b}{a} \right). \quad (5\text{-}8)$$

From (5-6) and (5-8), we obtain (for all a)

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right). \tag{5-9}$$

Example 5.2: $Y = X^2.$ (5-10)

$$F_Y(y) = P\big(Y(\xi) \le y\big) = P\big(X^2(\xi) \le y\big). \tag{5-11}$$

If $y < 0,$ then the event $\{X^2(\xi) \le y\} = \phi,$ and hence

$$F_Y(y) = 0, \quad y < 0. \tag{5-12}$$

For $y > 0,$ from Fig. 5.1, the event $\{Y(\xi) \le y\} = \{X^2(\xi) \le y\}$ is equivalent to $\{x_1 < X(\xi) \le x_2\}.$

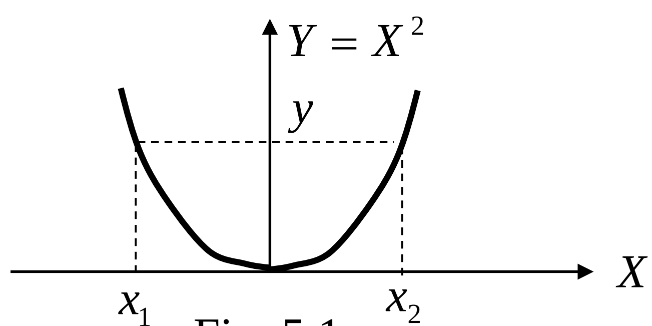

Fig. 5.1

Hence

$$F_Y(y) = P\big(x_1 < X(\xi) \le x_2\big) = F_X(x_2) - F_X(x_1)$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y}), \quad y > 0. \tag{5-13}$$

By direct differentiation, we get

$$f_Y(y) = \begin{cases} \dfrac{1}{2\sqrt{y}}\Big(f_X(\sqrt{y}) + f_X(-\sqrt{y})\Big), & y > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{5-14}$$

If $f_X(x)$ represents an even function, then (5-14) reduces to

$$f_Y(y) = \frac{1}{\sqrt{y}}\, f_X\big(\sqrt{y}\big)\, U(y). \tag{5-15}$$

In particular if $X \sim N(0,1)$, so that

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \tag{5-16}$$

and substituting this into (5-14) or (5-15), we obtain the p.d.f of $Y = X^2$ to be

$$f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-y/2} U(y). \tag{5-17}$$

On comparing this with (3-36), we notice that (5-17) represents a Chi-square r.v with $n = 1$, since $\Gamma(1/2) = \sqrt{\pi}.$ Thus, if X is a Gaussian r.v with $\mu = 0,$ then $Y = X^2$ represents a Chi-square r.v with one degree of freedom ($n = 1$).

Method 2

Note: As a general approach, given $Y = g(X)$, first sketch the graph $y = g(x)$, and determine the range space of y. Suppose $a < y < b$ is the range space of $y = g(x)$. Then clearly for $y < a$, $F_Y(y) = 0$, and for $y > b$, $F_Y(y) = 1$, so that $F_Y(y)$ can be nonzero only in $a < y < b$. Next, determine whether there are discontinuities in the range space of y. If so evaluate $P(Y(\xi) = y_i)$ at these discontinuities. In the continuous region of y, use the basic approach

$$F_Y(y) = P(g(X(\xi)) \le y)$$

and determine appropriate events in terms of the r.v X for every y. Finally, we must have $F_Y(y)$ for $-\infty < y < +\infty$, and obtain

$$f_Y(y) = \frac{dF_Y(y)}{dy} \quad \text{in} \quad a < y < b.$$

However, if $Y = g(X)$ is a continuous function, it is easy to establish a direct procedure to obtain $f_Y(y)$. A continuos function $g(x)$ with $g'(x)$ nonzero at all but a finite number of points, has only a finite number of maxima and minima, and it eventually becomes monotonic as $|x| \to \infty$. Consider a specific y on the y-axis, and a positive increment Δy as shown in Fig. 5.4

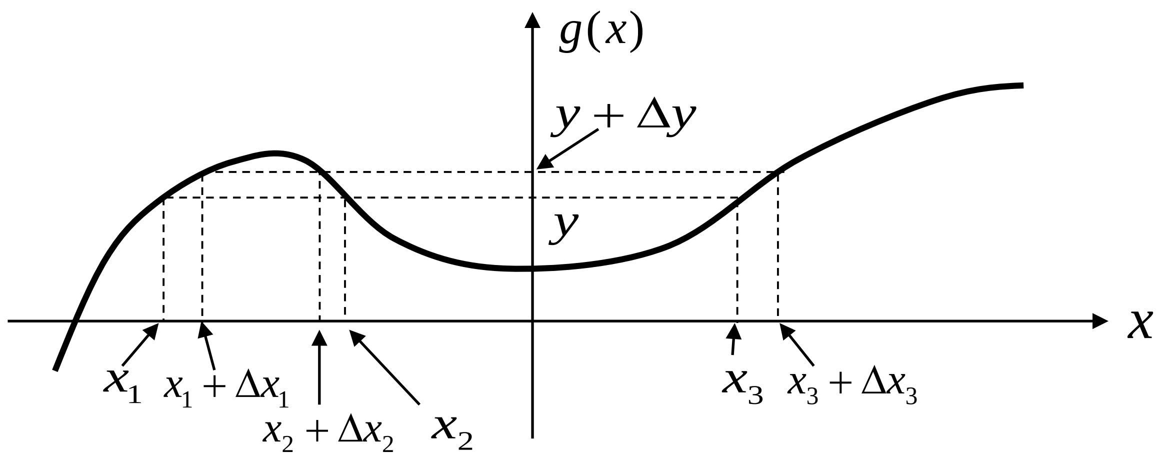

Fig. 5.4

$f_Y(y)$ for $Y = g(X)$, where $g(\cdot)$ is of continuous type.

Using (3-28) we can write

$$P\{y < Y(\xi) \le y + \Delta y\} = \int_{y}^{y+\Delta y} f_Y(u)\,du \approx f_Y(y) \cdot \Delta y. \qquad \text{(5-26)}$$

But the event $\{y < Y(\xi) \le y + \Delta y\}$ can be expressed in terms of $X(\xi)$ as well. To see this, referring back to Fig. 5.4, we notice that the equation $y = g(x)$ has three solutions x_1, x_2, x_3 (for the specific *y* chosen there). As a result when $\{y < Y(\xi) \le y + \Delta y\}$, the r.v *X* could be in any one of the three mutually exclusive intervals

$$\{x_1 < X(\xi) \le x_1 + \Delta x_1\},\ \{x_2 + \Delta x_2 < X(\xi) \le x_2\} \ \text{ or } \ \{x_3 < X(\xi) \le x_3 + \Delta x_3\}.$$

Hence the probability of the event in (5-26) is the sum of the probability of the above three events, i.e.,

$$P\{y < Y(\xi) \le y + \Delta y\} = P\{x_1 < X(\xi) \le x_1 + \Delta x_1\}$$
$$+ P\{x_2 + \Delta x_2 < X(\xi) \le x_2\} + P\{x_3 < X(\xi) \le x_3 + \Delta x_3\}. \qquad \text{(5-27)}$$

For small $\Delta y, \Delta x_i$, making use of the approximation in (5-26), we get

$$f_Y(y)\Delta y = f_X(x_1)\Delta x_1 + f_X(x_2)(-\Delta x_2) + f_X(x_3)\Delta x_3. \qquad (5\text{-}28)$$

In this case, $\Delta x_1 > 0$, $\Delta x_2 < 0$ and $\Delta x_3 > 0$, so that (5-28) can be rewritten as

$$f_Y(y) = \sum_i f_X(x_i)\frac{|\Delta x_i|}{\Delta y} = \sum_i \frac{1}{\left|\Delta y / \Delta x_i\right|} f_X(x_i) \qquad (5\text{-}29)$$

and as $\Delta y \to 0$, (5-29) can be expressed as

$$f_Y(y) = \sum_i \frac{1}{\left|dy/dx\right|_{x_i}} f_X(x_i) = \sum_i \frac{1}{\left|g'(x_i)\right|} f_X(x_i). \qquad (5\text{-}30)$$

The summation index *i* in (5-30) depends on *y*, and for every *y* the equation $y = g(x_i)$ must be solved to obtain the total number of solutions at every *y*, and the actual solutions $x_1, x_2, \cdots$ all in terms of *y*.

Examples

For example, if $Y = X^2,$ then for all $y > 0,$ $x_1 = -\sqrt{y}$ and $x_2 = +\sqrt{y}$ represent the two solutions for each *y*. Notice that the solutions x_i are all in terms of *y* so that the right side of (5-30) is only a function of *y*. Referring back to the example $Y = X^2$ (Example 5.2) here for each $y > 0,$ there are two solutions given by $x_1 = -\sqrt{y}$ and $x_2 = +\sqrt{y}.$ ($f_Y(y) = 0$ for $y < 0$).
Moreover

$$\frac{dy}{dx} = 2x \quad \text{so that} \quad \left|\frac{dy}{dx}\right|_{x=x_i} = 2\sqrt{y}$$

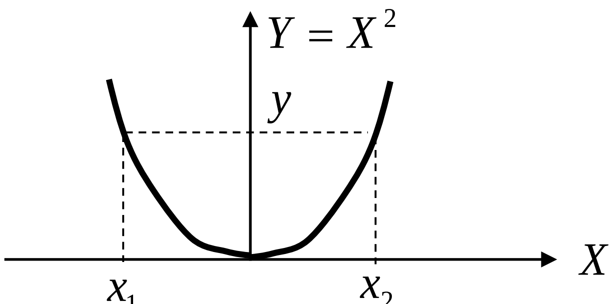

Fig. 5.5

and using (5-30) we get

$$f_Y(y) = \begin{cases} \dfrac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right), & y > 0, \\ 0, & \text{otherwise}, \end{cases} \qquad (5\text{-}31)$$

which agrees with (5-14).

Example 5.5: $Y = \dfrac{1}{X}$. Find $f_Y(y)$. (5-32)

Solution: Here for every *y*, $x_1 = 1/y$ is the only solution, and

$$\frac{dy}{dx} = -\frac{1}{x^2} \quad \text{so that} \quad \left|\frac{dy}{dx}\right|_{x=x_1} = \frac{1}{1/y^2} = y^2,$$

and substituting this into (5-30), we obtain

$$f_Y(y) = \frac{1}{y^2}\, f_X\!\left(\frac{1}{y}\right). \tag{5-33}$$

In particular, suppose *X* is a Cauchy r.v as in (3-39) with parameter α so that

$$f_X(x) = \frac{\alpha/\pi}{\alpha^2 + x^2}, \quad -\infty < x < +\infty. \tag{5-34}$$

In that case from (5-33), $Y = 1/X$ has the p.d.f

$$f_Y(y) = \frac{1}{y^2}\frac{\alpha/\pi}{\alpha^2 + (1/y)^2} = \frac{(1/\alpha)/\pi}{(1/\alpha)^2 + y^2}, \quad -\infty < y < +\infty. \tag{5-35}$$

But (5-35) represents the p.d.f of a Cauchy r.v with parameter $1/\alpha$. Thus if $X \sim C(\alpha)$, then $1/X \sim C(1/\alpha)$.

Example 5.6: Suppose $f_X(x) = 2x/\pi^2, \quad 0 < x < \pi$, and $Y = \sin X$. Determine $f_Y(y)$.

Solution: Since X has zero probability of falling outside the interval $(0, \pi)$, $y = \sin x$ has zero probability of falling outside the interval $(0,1)$. Clearly $f_Y(y) = 0$ outside this interval. For any $0 < y < 1$, from Fig.5.6(b), the equation $y = \sin x$ has an infinite number of solutions $\cdots, x_1, x_2, x_3, \cdots$, where $x_1 = \sin^{-1} y$ is the principal solution. Moreover, using the symmetry we also get $x_2 = \pi - x_1$ etc. Further,

$$\frac{dy}{dx} = \cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - y^2}$$

so that

$$\left| \frac{dy}{dx} \right|_{x = x_i} = \sqrt{1 - y^2}.$$

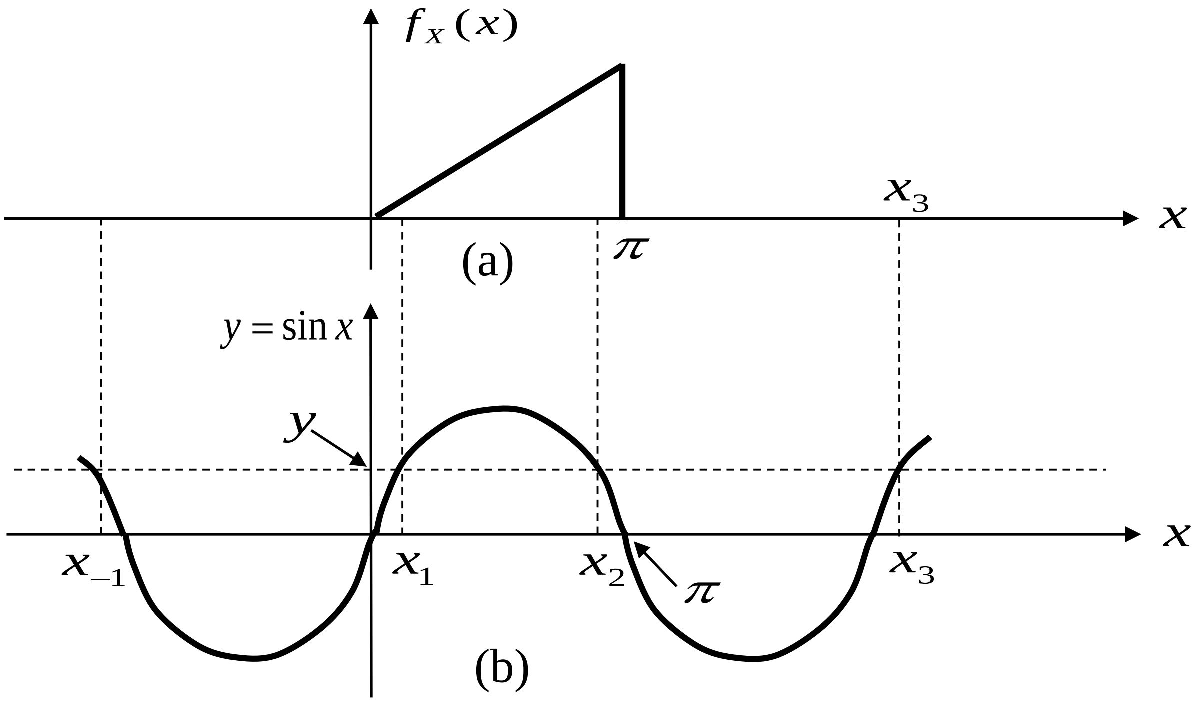

Fig. 5.6

Using this in (5-30), we obtain for $0 < y < 1$,

$$f_Y(y) = \sum_{\substack{i=-\infty \\ i \neq 0}}^{+\infty} \frac{1}{\sqrt{1-y^2}} f_X(x_i). \tag{5-36}$$

But from Fig. 5.6(a), in this case $f_X(x_{-1}) = f_X(x_3) = f_X(x_4) = \cdots = 0$ (Except for $f_X(x_1)$ and $f_X(x_2)$ the rest are all zeros).

Thus (Fig. 5.7)

$$f_Y(y) = \frac{1}{\sqrt{1-y^2}}\left(f_X(x_1) + f_X(x_2)\right) = \frac{1}{\sqrt{1-y^2}}\left(\frac{2x_1}{\pi^2} + \frac{2x_2}{\pi^2}\right)$$

$$= \frac{2(x_1 + \pi - x_1)}{\pi^2\sqrt{1-y^2}} = \begin{cases} \dfrac{2}{\pi\sqrt{1-y^2}}, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases} \qquad \text{(5-37)}$$

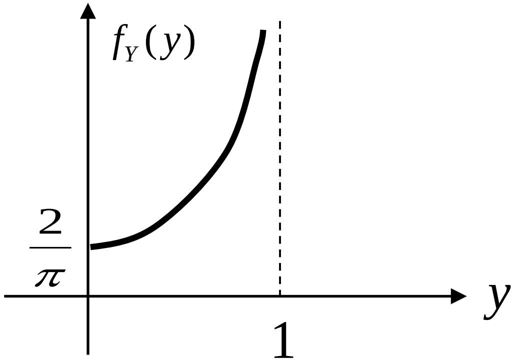

Fig. 5.7

Example 5.7: Let $Y = \tan X$ where $X \sim U(-\pi/2, \pi/2)$.
Determine $f_Y(y)$.

Solution: As *x* moves from $(-\pi/2,\ \pi/2)$, *y* moves from $(-\infty, +\infty)$.
From Fig.5.8(b), the function $Y = \tan X$ is one-to-one for $-\pi/2 < x < \pi/2$. For any *y*, $x_1 = \tan^{-1} y$ is the principal solution. Further

$$\frac{dy}{dx} = \frac{d \tan x}{dx} = \sec^2 x = 1 + \tan^2 x = 1 + y^2$$

so that using (5-30)

$$f_Y(y) = \frac{1}{|dy/dx|_{x=x_1}} f_X(x_1) = \frac{1/\pi}{1+y^2}, \quad -\infty < y < +\infty, \tag{5-38}$$

which represents a Cauchy density function with parameter equal to unity (Fig. 5.9).

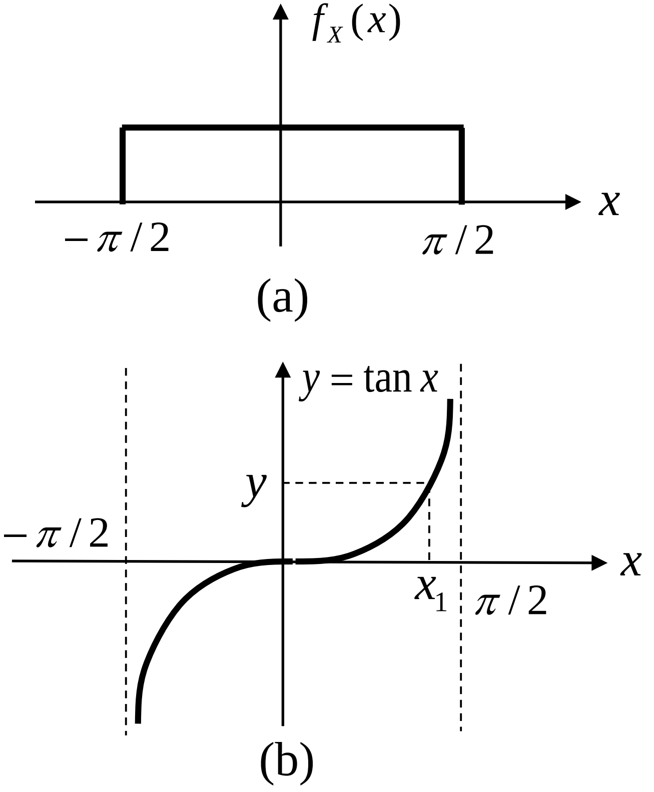

Fig. 5.8

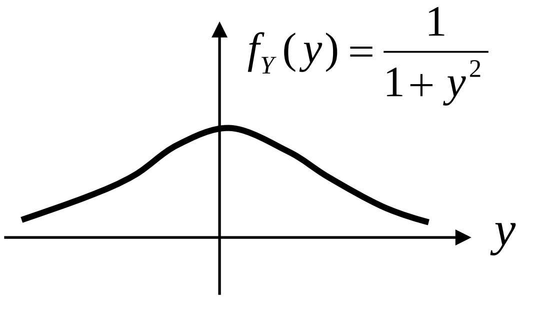

Fig. 5.9

Functions of a discrete-type r.v

Suppose X is a discrete-type r.v with

$$P(X = x_i) = p_i, \quad x = x_1, x_2, \cdots, x_i, \cdots \tag{5-39}$$

and $Y = g(X)$. Clearly Y is also of discrete-type, and when $x = x_i, \; y_i = g(x_i)$, and for those y_i

$$P(Y = y_i) = P(X = x_i) = p_i, \quad y = y_1, y_2, \cdots, y_i, \cdots \tag{5-40}$$

Example 5.8: Suppose $X \sim P(\lambda)$, so that

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0,1,2,\cdots \tag{5-41}$$

Define $Y = X^2 + 1$. Find the p.m.f of Y.

Solution: X takes the values $0,1,2,\cdots,k,\cdots$ so that Y only takes the value $1, 2, 5, \cdots, k^2 + 1, \cdots$ and

$$P(Y = k^2 + 1) = P(X = k)$$

so that for $j = k^2 + 1$

$$P(Y = j) = P\left(X = \sqrt{j-1}\right) = e^{-\lambda} \frac{\lambda^{\sqrt{j-1}}}{(\sqrt{j-1})!}, \quad j = 1, 2, 5, \cdots, k^2 + 1, \cdots. \qquad (5\text{-}42)$$

Example 5.3: Let

$$Y = g(X) = \begin{cases} X - c, & X > c, \\ 0, & -c < X \le c, \\ X + c, & X \le -c. \end{cases}$$

In this case

$$P(Y=0)=P(-c<X(\xi)\le c)=F_X(c)-F_X(-c). \qquad (5\text{-}18)$$

For $y>0$, we have $x>c$, and $Y(\xi)=X(\xi)-c$ so that

$$F_Y(y)=P\big(Y(\xi)\le y\big)=P(X(\xi)-c\le y)$$
$$=P\big(X(\xi)\le y+c\big)=F_X(y+c), \quad y>0. \qquad (5\text{-}19)$$

Similarly $y<0$, if $x<-c$, and $Y(\xi)=X(\xi)+c$ so that

$$F_Y(y)=P\big(Y(\xi)\le y\big)=P(X(\xi)+c\le y)$$
$$=P\big(X(\xi)\le y-c\big)=F_X(y-c), \quad y<0. \qquad (5\text{-}20)$$

Thus

$$f_Y(y)=\begin{cases} f_X(y+c), & y>0, \\ [F_X(c)-F_X(-c)]\delta(y), & \\ f_X(y-c), & y<0. \end{cases} \qquad (5\text{-}21)$$

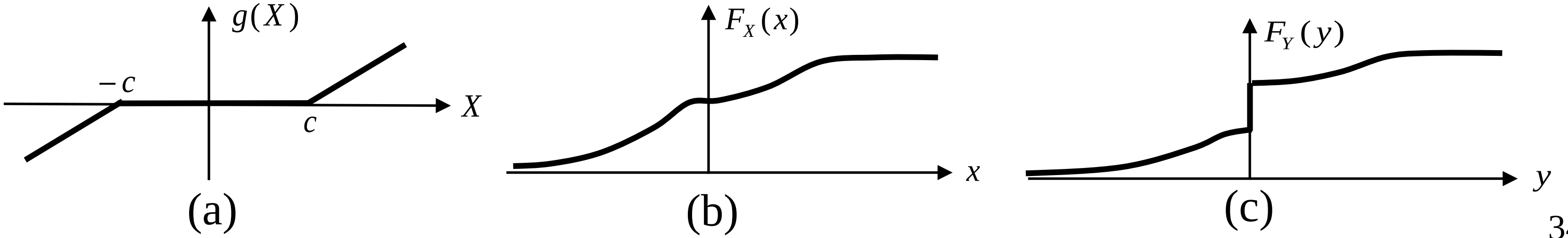

Fig. 5.2

Example 5.4: Half-wave rectifier

$$Y = g(X); \quad g(x) = \begin{cases} x, & x > 0, \\ 0, & x \le 0. \end{cases} \qquad (5\text{-}22)$$

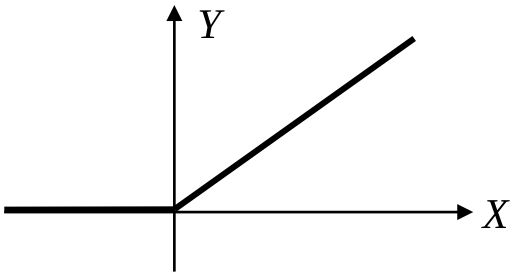

Fig. 5.3

In this case

$$P(Y = 0) = P(X(\xi) \le 0) = F_X(0). \qquad (5\text{-}23)$$

and for $y > 0$, since $Y = X$,

$$F_Y(y) = P\big(Y(\xi) \le y\big) = P\big(X(\xi) \le y\big) = F_X(y). \qquad (5\text{-}24)$$

Thus

$$f_Y(y) = \begin{cases} f_X(y), & y > 0, \\ F_X(0)\delta(y) & y = 0, \\ 0, & y < 0, \end{cases} = f_X(y)U(y) + F_X(0)\delta(y). \qquad (5\text{-}25)$$