Reminder of Probability

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OUTLINE:

- Outcomes and events;
- Definitions of probability;
- Probability algebra;
- Measure of dependence between events;

1.Outcomes and events

1.1. Outcomes of the experiment

Assume the following:

- we are given an experiment;
- outcomes are random;
- **example:** we are rolling a die.

Rolling a die:

- there are six possible outcomes: 6,5,4,3,2,1;
- the number which we get is outcome of the experiment.
- **A set of possible outcomes is called a sample space and denoted as:**

$$\Omega = \{1,2,3,4,5,6\} \qquad (1)$$

Note: sample space includes all simple results of the experiment.

1.2. Events

What is the event: 3 points of view

- 1- mathematically: event is a subset of set Ω ;
- 2- closer to practice: **the set of outcomes of the experiment**;
- 3- well known: phenomenon occurring randomly as a results of the experiment.

Difference between outcomes and events:

- outcomes: are given by the experiment itself;
- events: we can define events;
- simplest case: events and outcomes are the same.

Assume we have rolled a die:

- set of outcomes is:

$$\Omega = \{1,2,3,4,5,6\} \qquad (2)$$

Let us define the following events:

- event A_1 consisting in that we got 4 points:

$$A_1 = \{4\} \qquad (3)$$

- this events is the same as outcome 4 of the experiment.

- event A_2 consisting in that we got not less than 4 points:

$$A_2 = \{4,5,6\} \qquad (4)$$

- this event is different compared to outcomes.

- event A_3 consisting in that we got less than 3 points:

$$A_3 = \{1,2\} \qquad (5)$$

- this event is different compared to outcomes.

Notes:

- using the notion of outcomes we can define events;
- usually events and outcomes are different.

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1.3.Frequency-based computation of probability

Assume the following:

- we are given a fair die;
- meaning that we get one of the outcomes from subset Ω is 1/6.

Compute probabilities of events:

- event A_1 consisting in that we got 4 points:

$$\Pr\{A_1\} = \frac{1}{6} \qquad (6)$$

- event A_2 consisting in that we got not less than 4 points:

$$\Pr\{A_2\} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2} \qquad (7)$$

- event A_3 consisting in that we got less than 3 points:

$$\Pr\{A_3\} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3} \qquad (8)$$

Note: if we know probabilities of outcomes we can estimate probabilities of events.

1.4. Operations with events

Why we need it when events can be defined in terms of outcomes:

- we can also define events in terms of other events using set theory.

Let A and B be two sets of outcomes defining events:

- **union** of A and B is the following set:

$$A \cup B = \{x \in A \ \boldsymbol{OR} \ x \in B\} \qquad (9)$$

- **intersection** of A and B is the following set:

$$A \cap B = \{x \in A \ \boldsymbol{AND} \ x \in B\} \qquad (10)$$

- **difference** between A and B is the following set:

$$A - B = \{x \in A \ \boldsymbol{AND} \ x \neq B\} \qquad (11)$$

- **complement** of A is the set:

$$\bar{A} = \{x \in \Omega \ \boldsymbol{AND} \ x \neq A\} \quad (12)$$

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Operations can be graphically represented using Venn diagrams.

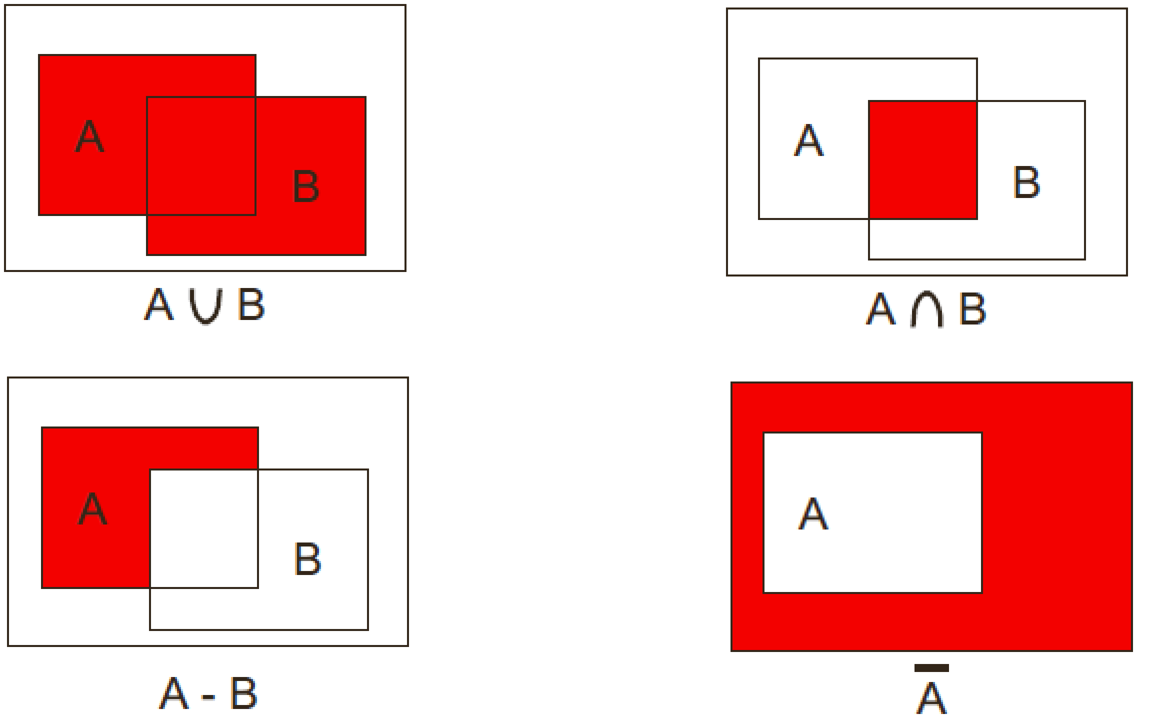

Figure 1: Graphical representation of operations with events.

1.5. Events using set theory

Why we need to use set theory:

- set theory provide a natural way to describe events.

Define the following:

- Ω is a set of all outcomes associated with an experiment:

- also called sample space;
- for a die the sample space is given by:

$$\Omega = \{1,2,3,4,5,6\} \qquad (13)$$

- $\mathcal{F}$ is a set of subsets (σ-algebra) of called events, such that:

- $\emptyset \in \mathcal{F}$ *and* $\Omega \in \mathcal{F}$
- $if\ A \in \mathcal{F} then$ the complementary set $\bar{A} \in \mathcal{F}$
- $if\ A_n \in \mathcal{F}, n = 1,2,\ldots, then\ \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

2.Denfitions of probability

2.1. Classic definition

Assume we have experiment E generating a set of events F such that:

- events are mutually exclusive: when one occurs, others do not occur!
- events generate a full group: $\bigcup_{i=1}^{n} A_i = \Omega$!
- events occur with equal chances: $\Pr\{A_1\} = \Pr\{A_2\} = \Pr\{A_n\}$!

Note: these events can be just outcomes.

Definition: outcome w favors event A, if w leads to A.

Definition: probability of A:

- ratio of the number of outcomes favoring A to all number of outcomes.

Example: rolling a die, events A consists in getting even number:

- number of outcomes favoring A: m = 3; all number of outcomes: n = 6;
- probability of A: $\Pr\{A\}$ = 3/6 = 1/2:

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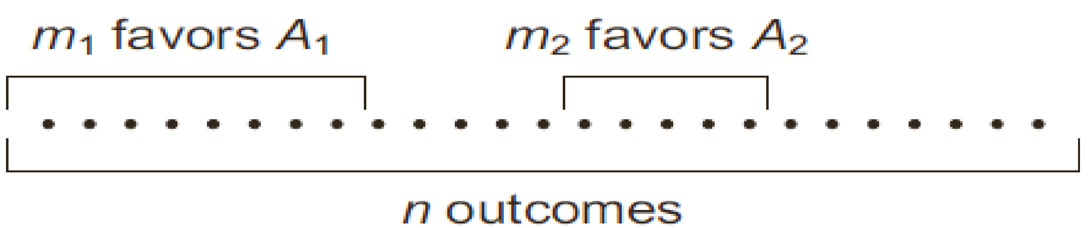
Properties of classic definition:

- assume we have n outcomes;
- all outcomes favor event $\Omega(\mathrm{m} = \mathrm{n})$: $\Pr\{\Omega\} = 1$;
- for any event out of F we have: $0 \leq \Pr\{A\} \leq 1, 0 \leq m \leq \mathrm{n}$;
- for complimentary event $\bar{A}$, we have:

$$\Pr\{\bar{A}\} = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - \Pr\{A\} \qquad (14)$$

- sum of exclusive events A_1 and A_2:

$$\Pr\{A_1 + A_2\} = \frac{m_1 + m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n} = \Pr\{A_1\} + \Pr\{A_2\} \qquad (15)$$



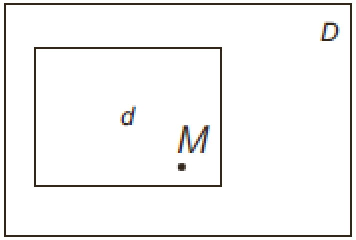
2.2. Geometric definition

Assume we have experiment E consisting in throwing a dot N into a space:

- space D is some space in R^N;
- dot is N-dimensional;
- probability of hitting any subspace d in D is equal.
- **Probability**: of hitting d is equal to:

$$\Pr\{M \in d\} = \frac{measure\ d}{measure\ D} \quad (16)$$

- measure here depends on R^N: if R^1 we can use length of D and d.



- Figure 2: Throwing a dot into space $D \in R^2$

2.3. Statistical definition

Why we need one more definition:

- classic and geometric definition have limited applicability;
- reason: it is not often possible to determine equally probable events.

Statistical definition:

- is known for centuries;
- stated by J. Bernoulli in his last work (1713);
- applicable to wide range of events: events with stable relative frequency.

Relative frequency of event A:

$$Pr^*\{A\} = \frac{\mu}{\mathrm{n}} \qquad (17)$$

- μ : number of experiments in which we observed A;
- n : whole number of experiments.

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Definition: probability of event A is the value to which $Pr^*\{A\}$ converges when $n \to \infty$.

$$\Pr\{A\} \overset{n\to\infty}{=} Pr^*\{A\} = \mu/n \quad (18)$$

Note:

- statistical definition have the same properties as classic one;
- the only method to compute approximate probabilities if the experiment is not classic.

If outcomes are equally likely to occur we can use two methods:

- classic computation using $\Pr\{A\}$ = m/n;
- using relative frequency: $\Pr\{A\} \overset{n\to\infty}{=} Pr^*\{A\} = \mu/n$

Button and Pearson compared number of heads in coin tossing:

$$n = 4040, \qquad \frac{m}{n} = 0.5, \qquad \frac{\mu}{n} = 0.5080$$

$$n = 12000, \qquad \frac{m}{n} = 0.5, \qquad \frac{\mu}{n} = 0.5016$$

$$n = 24000, \qquad \frac{m}{n} = 0.5, \qquad \frac{\mu}{n} = 0.5008$$

2.4. Axiomatic definition

THE EVENT SPACE :

- We need an event space which is rich enough to enable the computation of the probability for any event of practical interest.
- **Definition:** *A collection $\mathcal{F}$ of subsets of Ω is a **field** (or algebra) of subsets of Ω if the following properties are all satisfied:*

F1: $\emptyset \in \mathcal{F}$,
F2: *If* $A \in \mathcal{F}$ *then* $A^c \in \mathcal{F}$, *and*
F3: *If* $A_1 \in \mathcal{F}$ *and* $A_2 \in \mathcal{F}$ *then* $A_1 \cup A_2 \in \mathcal{F}$

2.4. Axiomatic definition

- Let $\mathcal{A}$ be a subset of P(Ω). Then the intersection of all σ-algebras containing $\mathcal{A}$ is a σ-algebras, called the smallest σ-algebras generated by $\mathcal{A}$. We denote the smallest σ-algebras generated by $\mathcal{A}$ by $\sigma(\mathcal{A})$.

Example : Smallest *σ*-field. The smallest σ-field associated with Ω is $F = \{\emptyset, \Omega\}$

Example : If A is a subset of Ω, then $F = \{\emptyset, A, \overline{A}, \Omega\}$ is a σ-field.

Example : Let $\Omega = \{a, b, c, d\}$.

A set $C = \{\{a\}, \{b\}\}$ is a subset of , but it is not a field.

Include:

$\{a\}^c = \{b, c, d\}$, $\{b\}^c = \{a, c, d\}$, $\{a\} \cup \{b\} = \{a, b\}$ and $(\{a\} \cup \{b\})^c = \{c, d\}$.

Thus the smallest σ-field containing all the elements of C *is:*

$\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{b, c, d\}, \{a, c, d\}, \Omega\}$

2.4. Axiomatic definition

- **Example:**

Let Ω={1, 2, 3, 4, 5, 6 } and $\mathcal{A}$={{1, 2}, {2, 3}}.

$\sigma(\mathcal{A})$ = { $\emptyset$, {1 , 2 , 3, 4 ,5, 6} ,
{1 , 2 } , { 3 , 4 , 5 , 6} ,
{ *2 , 3*} , {1 , 4 , 5, 6} ,
{1, 3, 4 , 5 , 6} , {*2*} , {2, 3, 4 , 5 , 6} , {1}, { 1 , 2 , 4 , 5 , 6} , {*3*} ,
{1, *2* , 3} , { 4 , 5 , 6} , {1 , 3} , {2, 4 , 5 , 6}}

Since: { 3 , 4 , 5 , 6} ∪ {1 , 4 , 5, 6} = {1, 3, 4 , 5 , 6} , {1, 3, 4 , 5 , 6}^{c}= {*2*} ,
{*2*} ∪ {3, 4 , 5 , 6}= {2, 3, 4 , 5 , 6} , {2, 3, 4 , 5 , 6}^{c}= {*1*} ,
{*2*} ∪ {1, 4 , 5 , 6}= {1, 2, 4 , 5 , 6} , {1,2, 4 , 5 , 6}^{c}= {3} ,
{1} ∪{3}={1,3}

2.4. Axiomatic definition

THE EVENT SPACE :

- **Example 2.** *In tossing a fair die*
- *(a) How many possible events are there?*
- *(b) Is the collection of all possible subsets of* Ω *a field?*
- *(c) Consider* $\mathcal{F} = \{\emptyset, \{1, 2, 3, 4, 5, 6\}, \{1, 3, 5\}, \{2, 4, 6\}\}$. *Is* $\mathcal{F}$ *a field?*

Solution.

(a) Using the Binomial Theorem, we find that there are $n = \sum_{k=0}^{6} C_{6,k} = (1 + 1)^6 = 64$ possible subsets of Ω. The number of possible events is 64 compared to only six possible outcomes.

Each of the collections (b) and (c) is a field, by checking F1, F2, and F3 ■

THE EVENT SPACE

F3a: If A_1, A_2, . . . are all in $\mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Definition 2.

*A collection $\mathcal{F}$ of subsets of Ω is a **sigma-field** (or **sigma-algebra**) of subsets of Ω*

if F1, F2, and F3a are all satisfied, where

Definition 3.

The pair $(\Omega, \mathcal{F})$ is called a measurable space.

2.4. Axiomatic definition

Some facts:

- the most accepted definition;
- includes classic, geometric and statistical as special cases;
- introduced by A.N. Kolmogorov in 1933.
- **Let Ω be the set of outcomes, $\mathcal{F}$ be σ-algebra:**
- P is a probability measure on $(\Omega, \mathcal{F})$ such that:
  - axiom 1: $\Pr\{A\} \geq 0$
  - axiom 2: $\Pr\{\Omega\} = 1$
  - axiom 3: $\Pr\{\phi\} = 0$
  - axiom 4: $\Pr\{\sum_k A_k\} = \sum_k \Pr\{A_k\}$ for mutually exclusive events.
  - (**another notation:** If $A_1, A_2, \ldots$ are mutually exclusive events in $\mathcal{F}$, then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$)

Note: P is the a mapping from $\mathcal{F}$ in [0, 1].

Definition 4. $(\Omega, \mathcal{F}, \mathrm{P})$ is called the probability space.

2.4. Axiomatic definition

More detail of Definition 4.

Definition 5. Measure Space: A triplet (Ω, $\mathcal{F}$, μ) is a measure space if (Ω, $\mathcal{F}$) is a measurable space and μ: $\mathcal{F} \rightarrow [0; \infty)$ is a measure.

Definition 6. Probability Space: A measure space is a probability space if μ (Ω)=1. In this case, μ is a probability measure, which we denote P.

Let P be a probability measure. The *cumulative distribution function* (c.d.f.) of P is defined as: $F(x) = P((-\infty, x])$, $x \in R$

2.4. Axiomatic definition

Question

- What is Union Bound?

2.4. Axiomatic definition

Example : A fair die is tossed once. What is the probability of an even number occurring?

A: The sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$.
We assign a weight of *w* to each sample point; i.e., $P(i) = w$, $i = 1, 2, \ldots, 6$. By the 2nd and 4th axioms we have $P(\Omega) = 1 = 6w$; hence, $w = 1/6$. Letting $A = \{2, 4, 6\}$, $P(A) = P(\{2\}) + P(\{4\}) + P(\{6\}) = 1/6 + 1/6 + 1/6 = 1/2$

2.4. Axiomatic definition

Example: The sample space of a die is $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{F} = \mathcal{P}(\Omega)$, where $\mathcal{P}$ is the *power set* of Ω. The probability measure μ is completely determined by the values of $\mu\{1\}, \mu\{2\}, \ldots, \mu\{6\}$.

For example, suppose:

$\mu\{1\} = 1/12$	$\mu\{4\} = 1/6$
$\mu\{2\} = 1/12$	$\mu\{5\} = 1/6$
$\mu\{3\} = 1/3$	$\mu\{6\} = 1/6$

Then the probability of rolling a 2 or a 3 is $\mu\{2, 3\} = 1/12 + 1/3 = 5/12$.

3. Probability algebra

3.1. Adding events

For mutually exclusive events:

$$\Pr\{\textstyle\sum_k A_k\} = \textstyle\sum_k \Pr\{A_k\} \qquad (20)$$

- holds only when events are exclusive!!!

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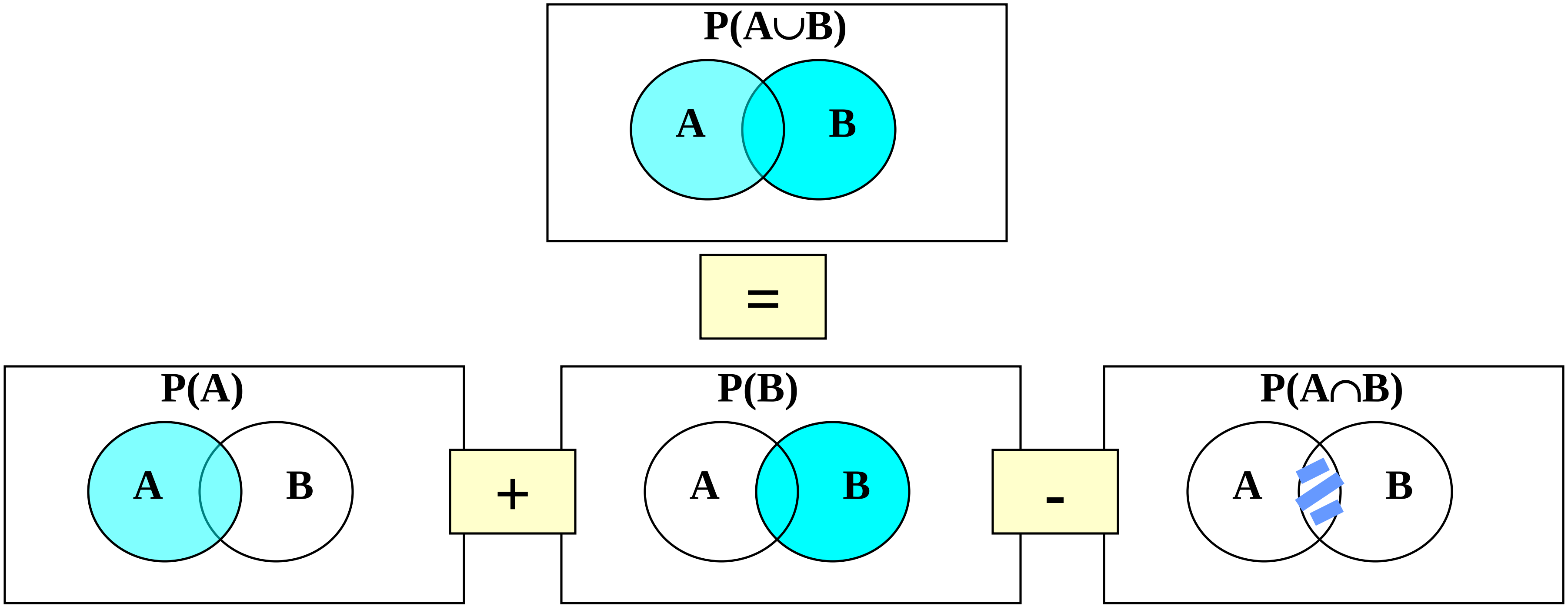
3. Probability algebra

3.1. Adding events

For two arbitrary events:

$$\Pr\{A + B\} = \Pr\{A\} + \Pr\{B\} - \Pr\{AB\} \qquad (21)$$

Thus $\Pr\{A + B\} \leq \Pr\{A\} + \Pr\{B\}$



3. Probability algebra

3.1. Adding events

For three arbitrary events: principle of inclusion-exclusion

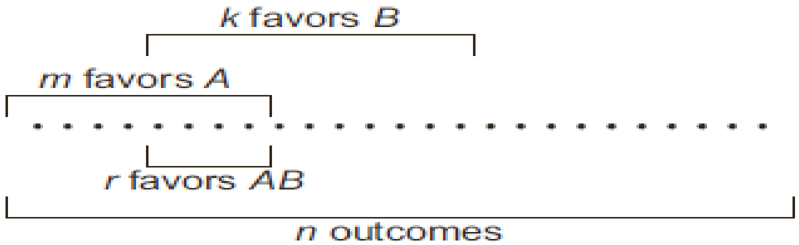
$$\Pr\{A + B + C\} = \Pr\{A\} + \Pr\{B\} - \Pr\{AB\} - \Pr\{BC\} - \Pr\{AC\} + \Pr\{ABC\} \qquad (22)$$

Note: one can extend it to n events.

3.2. Conditional probability

Definition: probability that the event A will occur given that the event B has already occurred.

Consider the classic experiment with equally probable outcomes:



- probabilities: $Pr\{A\} = m/n$, $Pr\{B\} = k/n$, $Pr\{AB\} = r/n$;
- if event B already occurred then for event A the number of outcomes decreases to k;
- among k there are r outcomes favoring A: using classic definition $Pr\{A|B\} = r/k$;
- dividing nominator and denominator by n we get:

$$Pr\{A|B\} = \frac{r}{k} = \frac{r/n}{k/n} = \frac{Pr\{AB\}}{Pr\{B\}} \qquad (23)$$

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Result:

$$Pr\{A|B\} = \frac{Pr\{AB\}}{Pr\{B\}} \qquad (24)$$

Note the following:

- one can check that $Pr\{.|B\}$ is a probability measure;
- the conditional probability is not defined if $Pr\{B\}$ = 0.

Note:

- Event B with prob. 0 (i.e. $Pr\{B\}$ = 0) is different from
- Impossible event (i.e. $Pr\{\phi\}$=0)

We can change the role of A and B:

$$Pr\{B|A\} = \frac{Pr\{AB\}}{Pr\{A\}} \qquad (25)$$

Useful notation:

$$p_{X,Y}(x, y \mid A) = \begin{cases} \dfrac{p_{X,Y}(x, y)}{P\{A\}} & (x, y) \in A, \quad P\{A\} > 0 \\ 0 & \text{otherwise} \end{cases} \qquad (2.5)$$

- **Example**: there are three trunks:
- we seize the trunk with probability 1/3;
- what is the probability to seize a given trunk in two attempts (at least once)?
- let: A: we seize the trunk in first attempt, B: we seize the trunk in second attempt:

$$\Pr\{A\} + \Pr\{B\} - \Pr\{AB\} = \frac{1}{3} + \frac{1}{3} - \frac{1}{9} = \frac{5}{9} \qquad (26)$$

put it in another way (first attempt and not the second one (i.e. $\frac{1}{3} * \frac{2}{3}$+ vice versa (i.e . $\frac{2}{3} * \frac{1}{3}$)+ both (i.e. . $\frac{1}{3} * \frac{1}{3}$)

- Or:

1 − prob. (we do not seize the trunk in both attempts)

$$1 - (1 - \Pr\{A\})(1 - \Pr\{B\}) = 1 - \left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{3}\right) = 1 - \frac{4}{9} = \frac{5}{9}$$

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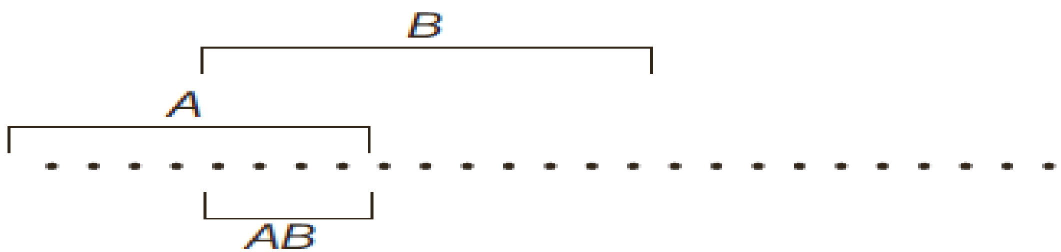
3.3. Multiplication of events

- Multiplication of two arbitrary events:

$$\Pr\{AB\} = \Pr\{B\}\ \Pr\{A|B\} = \Pr\{A\}\ \Pr\{B|A\} \qquad (27)$$



- Multiplication of n arbitrary events:

$$\Pr\{A_1A_2 \dots A_n\} = \Pr\{A_1\}\ \Pr\{A_2|A_1\}\Pr\{A_3|A_1A_2\}\dots\Pr\{A_n|\,A_1 \dots A_{n-1}\} \qquad (28)$$

verification:

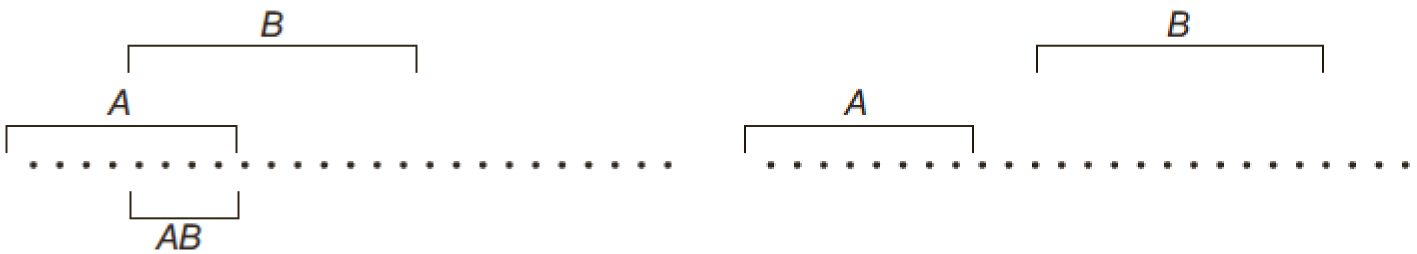
$$\Pr\{A_1A_2 \dots A_n\} = \Pr\{A_1A_2\}\Pr\{A_3|A_1A_2\}\dots\Pr\{A_n|\,A_1 \dots A_{n-1}\}$$

$$\Pr\{A_1A_2 \dots A_n\} = \Pr\{A_1A_2A_3\}\dots\Pr\{A_n|\,A_1 \dots A_{n-1}\}$$

3.4. Independent and dependent events

Definition: event A is independent of event B if the following holds:

$$Pr\{A|B\} = Pr\{A\} \qquad (29)$$



Note: if A is independent of B then B is independent of A.

Multiplication: if events A and B are independent then their product:

$$\Pr\{AB\} = \Pr\{B\}\ \Pr\{A|B\} = \Pr\{A\}\ \Pr\{B\} \qquad (30)$$

Note: $\Pr\{AB\} = \Pr\{A\}\ \Pr\{B\}$is sufficient for events to be independent.

3.4. disjoint events

Definition: If $E1 \cap E2 = \emptyset$, then $E1$ and $E2$ are ***pairwise exclusive (disjoint) note:***

Note: ***pairwise exclusive*** implies ***mutually (collectively) exclusive (disjoint).***

Definition: If $E_1, E_2, \ldots, E_n$ are events such that $E_i \cap E_j = \emptyset,\ \forall i, j$, and such that $\bigcup_{i=1}^{n} E_i = \Omega$, then we say that events $\boldsymbol{E_1, E_2, \ldots, E_n}$ ***partition*** set Ω.

$$
\begin{array}{cc}
 & \begin{array}{cccccc} & E_1 & & & & \end{array} \\
\Omega = & \left\{ \begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\
(6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)
\end{array} \right.
\end{array}
\qquad E_2
$$

Events E_1 and E_2 in sample space Ω

Later: ***pairwise independence*** does not imply ***mutually (collectively) independence.***

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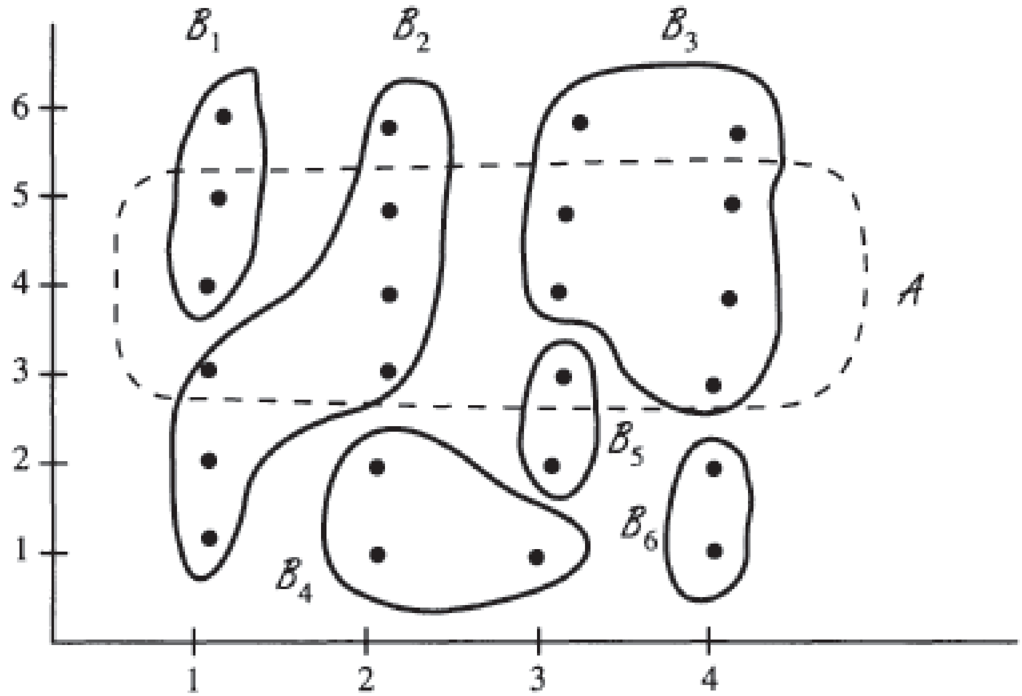
3.4. disjoint events



Partition of a Set of Pairs

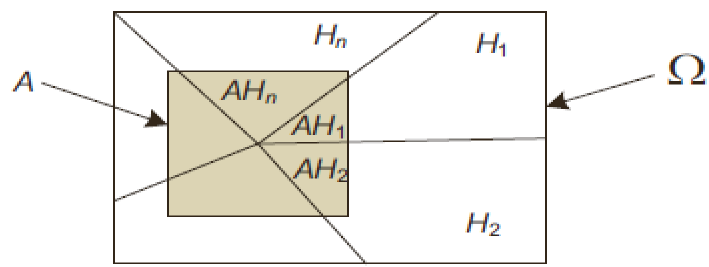
3.5. Law of total probability

Let $H_1, H_2, \dots, H_n$ be the events such that:

- $H_i \cap H_j = 0 \; if \; i \neq j$ (mutually exclusive events);
- $\Pr\{H_i\} > 0 \; for \; i = 1,2,\dots,n$ (non-zero);
- $H_1 \cup H_2 \cup \cdots \cup H_n = \Omega$ (full group)

Then for any event A the following result holds:

$$\mathrm{A} = \mathrm{A} \cap \Omega = A(H_1 \cup H_2 \cup \cdots \cup H_n) = AH_1 \cup AH_2 \cup \cdots + \cup AH_n = \cup_{i=1}^{n} AH_i \qquad (31)$$



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We have:

$$Pr\{A\} = \Pr\{\sum_{i=1}^{n} AH_i\} = \sum_{i=1}^{n} \Pr\{AH_i\} = \sum_{i=1}^{n} \Pr\{A|H_i\} \Pr\{H_i\} \quad (32)$$

Law of total probability:

$$Pr\{A\} = \sum_{i=1}^{n} \Pr\{A|H_i\} \Pr\{H_i\} \quad (33)$$

Notes:

- events H_i, i = 1, 2,…,n are called hypotheses;
- probabilities $\Pr\{H_1\}, \Pr\{H_2\}, \dots, \Pr\{H_n\}$ are called apriori probabilities.

Law of total probability helps to find probability of event A if we know:

- probabilities of hypotheses H_i,i = 1,2,…, n;
- probabilities $\Pr\{A|H_i\}$.

Example: similar components are made by 3 vendors, we have:

- vendor 1: 50% of components: probability of non-conformance is 0:002;
- vendor 2: 30% of components: probability of non-conformance is 0:004;
- vendor 3: 20% of components: probability of non-conformance is 0:005.

Question: if we take 1 component what is the probability that it is non-conformant.

- H_k the chosen detail is made by vendor k = 1,2,3;

$$Pr\{H_1\} = 0.5\,, \Pr\{A|H_1\} = 0.002,$$
$$Pr\{H_2\} = 0.3\,, \Pr\{A|H_2\} = 0.004,$$
$$Pr\{H_3\} = 0.2\,, \Pr\{A|H_3\} = 0.005.$$

- A: the chosen component is non-conformant.

Using the law of total probability:

$$Pr\{A\} = \Pr\{H_1\}\Pr\{A|H_1\} + \Pr\{H_2\}\Pr\{A|H_2\} + \Pr\{H_3\}\Pr\{A|H_3\} = 0.0032 \quad (35)$$

3.6. Bayes' formula

Assume: we carried out experiment and event A occurred:

- we have to re-evaluate probabilities of hypotheses: $\Pr\{H_1\}, \Pr\{H_2\}, \ldots, \Pr\{H_n\}$;
- we are looking for $\Pr\{H_1|A\}, \Pr\{H_2|A\}, \ldots, \Pr\{H_n|A\}$;

Use formula for conditional probability $\Pr\{A|B\} = \frac{\Pr\{AB\}}{\Pr\{B\}}$ **get:**

$$\Pr\{H_k|A\} = \frac{\Pr\{AH_k\}}{\Pr\{A\}} = \frac{\Pr\{H_k\}\Pr\{A|H_k\}}{\Pr\{A\}} \qquad (36)$$

Use law of total probability $\Pr\{A\} = \sum_{i=1}^{n} P\{A|H_i\}\Pr\{H_i\}$ **to get:**

$$\Pr\{H_k|A\} = \frac{\Pr\{H_k\}\Pr\{A|H_k\}}{\sum_{i=1}^{n} P\{A|H_i\}\Pr\{H_i\}} \quad k = 1,2, \ldots, n \qquad (37)$$

- this formula is known as Bayes's formula.

Note: Probabilities $\Pr\{H_1|A\}, \Pr\{H_2|A\}, \ldots, \Pr\{H_n|A\}$ are called aposteriori probability.

Example: similar components are made by 3 vendors, we get

- vendor 1: 50% probability of non-conformance is 0:002;
- vendor 2: 30% probability of non-conformance is 0:004;
- vendor 3: 20% probability of non-conformance is 0:005;
- if we take 1 component, the probability that it is non-conformant:

$$Pr\{A\} = \Pr\{H_1\}\Pr\{A|H_1\} + \Pr\{H_2\}\Pr\{A|H_2\} + \Pr\{H_3\}\Pr\{A|H_3\} = 0.0032 \qquad (38)$$

Question: we took 1 component and it is non-conformant, which vendor to blame?

$$\Pr\{H_1|A\} = \frac{Pr\{H_1\}\Pr\{A|H_1\}}{\Pr\{A\}} = \frac{5}{16}$$

$$\Pr\{H_2|A\} = \frac{Pr\{H_2\}\Pr\{A|H_2\}}{\Pr\{A\}} = \frac{6}{16}$$

$$\Pr\{H_3|A\} = \frac{Pr\{H_3\}\Pr\{A|H_3\}}{\Pr\{A\}} = \frac{5}{16}$$

Answer: most probably vendor 2.

3.7. Measure of dependence between events

If events A and B are dependent:

- we can measure the dependence as:

$$Pr\{A|B\} = \frac{Pr\{AB\}}{Pr\{B\}},\ Pr\{B|A\} = \frac{Pr\{AB\}}{Pr\{A\}} \qquad (43)$$

- **the problem:** these metrics are not symmetric.

Symmetric measure of dependence between events:

$$\rho_{AB} = \frac{\Pr\{AB\} - \Pr\{A\}\Pr\{B\}}{\sqrt{\Pr\{A\}\Pr\{\bar{A}\}\Pr\{B\}\Pr\{\bar{B}\}}} \qquad (44)$$

Properties of ρ_{AB} :

- $\rho_{AB} = 0$ when and only when A and B are independent;
- $-1 \le \rho_{AB} \le 1$;
- $\rho_{AB} = 1$ when $Pr\{A\} = Pr\{B\} = Pr\{AB\}$, $\rho_{AB} = -1$ when $A = \bar{B}$;
- $\rho_{\bar{A}B} = -\rho_{AB}$;